

A Framework For Designing Information Elicitation Mechanisms That Reward Truth-telling

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Abstract

Information elicitation mechanisms, such as Peer Prediction [11] and Bayesian Truth Serum [12], are designed to reward agents for honestly reporting their private information, even when this information cannot be directly verified. Information elicitation mechanisms, such as these, are cleverly designed so that truth-telling is a strict Bayesian Nash Equilibrium. However, a key challenge that has arisen is that there are typically many other non-truthful equilibria as well, and it is important that truth-telling not only be an equilibrium, but be paid more than other equilibria so that agents do not coordinate on a non-informative equilibria. Several past works have overcome this challenge in various settings using clever techniques, but a new technique was required for each new setting.

Our main contribution in this paper is providing a framework for designing information elicitation mechanisms where truth-telling is the highest paid equilibrium, even when the mechanism does not know the common prior. We define *information monotone functions* which can measure the amount of “information” contained in agents’ reports such that the function is greater in the truthful equilibrium than in non-truthful equilibria. We provide several interesting information monotone functions (f -disagreement, f -mutual information, f -information gain) in different settings. Aided by these theoretical tools, we (1) extend Dasgupta and Ghosh [2]’s mechanism to the non-binary setting with an additional assumption that agents are asked to answer a large number of a priori similar questions; (2) reprove the main results of Prelec [12], Dasgupta and Ghosh [2] and a weaker version of Kong and Schoenebeck [9] in our framework. Our framework thus provides both new mechanisms and a deeper understanding of prior results.

1 Introduction

In the information age, the wisdom of the crowd becomes more and more accessible. Many social information sites (e.g. Quora, Yahoo!) help users solve questions by aggregating a large group’s answers. Online retailers (e.g. Amazon, Ebay) collect customers’ reviews to improve new customers’ shopping experiences. However, a failure of people perceiving value in providing their information leads to low participation rates which may lead to unrepresentative feedback or the lack of feedback altogether. Even with rewards, agents may not have enough motivation to expend effort in answering questions, especially when they are rewarded equally once they participate. Moreover, sometimes people may lie for potential future benefit. Thus the systems that do not reward agents or reward agents equally may collect less representative, meaningless and even misleading answers. To collect more representative answers, systems that can motivate honesty should be developed.

Honesty can easily be motivated when answers can be verified. For example, when students take exams, they have a strong motivation to tell the truth since they know the graders know the ground truth of each question and will grade the exams based on the ground truth. However, in some situations, the objective truth is not observable or even does not exist.

To motivate honesty without verification, Miller et al. [11] propose a mechanism: *peer prediction* that is based on a clever insight: every agent’s information is related to her peers’ information and therefore can be checked using her peers’ information. Peer prediction is more powerful than a simple majority vote, and can motivate agents who believe they are in the minority to report truthfully. Several additional mechanisms have been proposed for two main settings: when each agent provides answers to multiple similar questions (e.g. [2]), and when each agent provides one answer (e.g. [11, 12, 20, 13, 14]). The typical solution desiderata is that truth-telling is a strict Bayesian Nash equilibrium.

When truth-telling is a strict Bayesian Nash equilibrium, if an agent believes most other agents will report honestly, then she will tell the truth as well. However, in many information elicitation mechanisms, additional non-truthful equilibrium exist as well. Moreover, some of these equilibrium may pay more than truth-telling (in expectation) and require little effort. In such cases, agents may believe that other agents will play these alternate equilibria, and do so as well. We can think of this as the agents *ex ante* “colluding” with each other to choose an “effortless” equilibrium which leads to meaningless reports. For example, when students are asked to grade each other for their homework, it is very likely that they may just give good grades without reading the other students’ work—both decreasing effort and increasing agreement. Two additional examples would be agents simply choosing the answer they think will be most common, or simply choosing the value with the highest possible payoff. Recent research [4] indicates that individuals in lab experiments do not always truth-tell when faced with peer prediction mechanisms; this may in part be related to the issue of equilibrium multiplicity. Thus, it is important to design mechanisms that discourage non-truthful equilibria.

While prior works (see Section 2) deal with the equilibrium multiplicity issue, their results are typically proved by clever algebraic computations and sometimes lack a deeper intuition. In this paper, we provide a framework that helps give a deeper understanding of the equilibrium multiplicity issue via tools based on information theory. We use this framework to show some interesting new results as well as to give more systematic proofs of three old results.

Our Contributions

1. We provide an information monotone framework that creates mechanisms where truth-telling has the maximal expected payoff among all equilibrium (Section 3).
2. We define *information monotone functions* which can measure the amount of “information” contained in agents’ reports such that the function is greater in the truthful equilibrium than in non-truthful equilibria. We provide several information monotone functions (f -disagreement, f -mutual information, and f -information gain) for different settings. Aided by these information monotone functions, we study two different settings:
 - (a) **The multiple questions setting:** In this setting (see Section 4) agents are required to answer multiple similar questions, each with a finite answer space. In this setting, we use f -mutual information (Section 5.1) to create an f -mutual information mechanism (Section 5.2) that has truth-telling as a Bayesian Nash equilibrium when the number of questions is large. Moreover, this truth-telling equilibrium pays agents more than other equilibria. This extends the results of Dasgupta and Ghosh [2] to the non-binary setting. We also use f -mutual information to reprove (Section 5.3) the main results of Dasgupta and Ghosh [2] (which apply to the binary answer case, but do not require a large number of questions).

- (b) **The one question setting:** In this setting (see Section 6) agents are required to answer one question; however there is a necessary additional assumption that agents share a common prior. First (Sections 7), we use f -disagreement to design a mechanism where truth-telling is a strict Bayesian Nash equilibrium, and both pays more than other symmetric equilibria, and, if the number of agents is sufficiently large, pays more than any equilibrium. However, this mechanism requires the additional assumption on the prior of “self-predicting” signals. This is a weaker results than that of Kong and Schoenebeck [9], but is also substantially simpler in both the mechanism and the analysis. Second, we reinterpret (and reprove) the main results of Bayesian Truth Serum (BTS) [12] via f -information gain (Section 8). In comparison to the previous results, here a large number of agents are required just to make truth-telling an equilibrium. Bayesian Truth Serum is a very important work in the peer-prediction literature, and we believe our framework highlights important insights of BTS which can be extended to other settings.

We also give an impossibility result which illustrates, in a certain sense, the tightness of the results on the f -mutual information mechanism (Section 5.2). This extends the work of Kong and Schoenebeck [9], which shows the optimality of the mechanism designed in Section 7.3.

2 Related Work

There are several works [12, 6, 7, 2, 8, 15] that focus on designing mechanisms that discourage non-truthful equilibria after Miller et al. [11] introduced peer prediction.

Prelec [12] proposes Bayesian Truth Serum (BTS) and the signal-prediction framework for the setting that agents are asked to answer only one question and the mechanism does not know the common prior. Prelec [12] shows when the number of agents is infinite, the case everyone tells the truth is an equilibrium and each agent obtains the highest *information score* if everyone tells the truth. Radanovic and Faltings [15] consider the setting of sensors and they do not require sensors to report a prediction. Instead of comparing each sensor’s prediction to the distribution of all sensors’ information like BTS, they compare the distribution of each sensor’s local peers’ information and the distribution of all sensors’ information. This work deals with the equilibrium multiplicity issue in the same manner as BTS. Prelec [12]’s signal-prediction framework also inspired a host of results [20, 13, 14, 23, 16, 3, 21, 22, 19] about designing mechanisms without a known common prior; however, these works lack the analysis of non-truthful equilibria.

The biggest limitation of Prelec [12] is that the number of agents is assumed to be infinite even to make truth-telling an equilibrium. Witkowski and Parkes [20], Radanovic and Faltings [13, 14], Zhang and Chen [23], Riley [16], Faltings et al. [3] successfully weaken this assumption. In particular, Radanovic and Faltings [13] proposed a specific family of mechanisms: Decomposable Payment Schemes (DPS) and show that, with an additional assumption on the prior, there exists a mechanism called Multi-valued RBTS in DPS that has truth-telling as an equilibrium for small group of agents and non-binary signals. However, all of these works lack the analysis of non-truthful equilibria. Kong and Schoenebeck [9], in a prior work, propose a mechanism called the Disagreement Mechanism and solve the equilibrium multiplicity issue via tools from information theory. Moreover, the Disagreement Mechanism does not assume the additional assumption about the prior Multi-valued RBTS requires and can be applied to a small group of agents and non-binary signals. Kong and Schoenebeck [9] also shows the impossibility of making truth-telling pay strictly more than a certain class of equilibria. The impossibility result in the current paper generalizes

Kong and Schoenebeck [9] to when there is no guarantee that the prior is symmetric. Our present work both generalize the tools of Kong and Schoenebeck [9] and provides a more general application.

Jurca and Faltings [5, 7] analyse non-truthful *pure* strategies of peer-prediction when the mechanism knows the prior and leave the analysis of mixed strategies as an open question. Kong et al. [10] analyze all equilibria including truthful and non-truthful, pure and mixed strategies of the peer-prediction mechanism. Additionally, they “optimize” the mechanism over a natural space. However, unlike the current work, the mechanism still needs to know the prior and the analysis only works for the case of binary signals.

Dasgupta and Ghosh [2] consider a different setting where agents are asked to answer multiple a priori similar binary questions. Dasgupta and Ghosh [2] propose a mechanism M_d that pays each agent the correlation between her answer and her peer’s answer. Dasgupta and Ghosh [2] show each agent obtains the highest payment if everyone tells the truth. In retrospect, one can see that our techniques are a recasting and generalization of those of Dasgupta and Ghosh [2]. Kamble et al. [8] also considers the setting of multiple a priori similar questions. Their mechanism applies to both homogeneous and heterogeneous populations. Agents are required to answer a large number of a priori similar questions. However, their mechanism contains non-truthful equilibria that are paid greater than truth-telling.

2.1 Independent Work

Like the current paper, Shnayder et al. [17] also extends Dasgupta and Ghosh [2]’s binary signals mechanism to multiple signals setting. However, the two works differ both in the specific mechanism and the technical tools employed.

Shnayder et al. [17] analyze how many questions are needed (whereas we simply assume infinitely many questions). Like our paper, they also analyze to what extent truth-telling can pay strictly more than other equilibria. Additionally, they show their mechanism does not need a large number of questions when “the signal correlation structure” is known (that is the pair-wise correlation between the answers of two questions). While, the current paper does not state such results; we note that the techniques employed are sufficiently powerful to immediately extend to this interesting special case: when the signal structure is known, it is possible to construct an unbiased estimator for f -mutual information of the distribution, when the total variation distance is used to define the f -mutual information. Both papers also show their results generalize Dasgupta and Ghosh [2]’s.

3 Monotone Information Framework

3.1 General Setting

We introduce the general setting (n, \mathcal{PI}) of the information monotone framework where n is the number of agents and \mathcal{PI} is the set of possible private information. Each agent i receives private information $PI_i \in \mathcal{PI}$. She also has a prior for other agents’ private information. Formally, each agent i believes the agents’ private information is chosen from a joint distribution Q_i before she receives her private information. Thus, from agent i ’s prospective, before she receives any private information, the probability that agent 1 receives PI_1 , agent 2 receives PI_2 , ..., agent n receives PI_n is $Pr_{Q_i}(PI_1, PI_2, \dots, PI_n)$. After she receives her private information based on her prior, agent i will also update her knowledge to a posterior distribution which is the prior conditioned on her private information. We define $\Delta_{\mathcal{PI}}$ as the set of all possible probability distributions over \mathcal{PI} . We also define $\mathbf{Q} = (Q_1, Q_2, \dots, Q_n)$ as a prior profile where Q_i is agent i ’s prior.

Now we give two definitions that describe the prior:

Definition 3.1 (Common Prior). *We say agents have common prior if it is a common knowledge that agents' private signals for the question are chosen from a common joint distribution Q over \mathcal{PI}^n .*

Definition 3.2 (Symmetric Prior). *We say the prior Q is symmetric if for any permutation $\pi : [n] \mapsto [n]$,*

$$Pr_{Q_i}(PI_{\pi(1)}, PI_{\pi(2)}, \dots, PI_{\pi(n)}) = Pr_{Q_i}(PI_1, PI_2, \dots, PI_n)$$

Example 3.3. *We say a biased coin has parameter p if with probability p , it comes up heads; with probability $1 - p$, it comes up tails. Consider a coin with parameter p uniformly distributed between $2/5$ and $4/5$. Agents receive private information via flipping the biased coin once independently.*

In this case, $\mathcal{PI} = \{0, 1\}$. Agents share symmetric common prior: $Pr(1, 1) = \int_{2/5}^{4/5} p^2 dp$, $Pr(0, 0) = \int_{2/5}^{4/5} (1 - p)^2 dp$, $Pr(1, 0) = Pr(0, 1) = \int_{2/5}^{4/5} p(1 - p) dp$.

The prior tells the agent (i) the probability she will receive heads is $q(1) = \frac{3}{5}$, (ii) once she receives heads, the probability another agent receives heads would be $q(1|1) = \frac{28}{45}$; (iii) once she receives tails, the probability another agent receives heads would be $q(1|0) = \frac{17}{30}$.

Ideally, a mechanism will reward agents for accurately reporting their private information.

Definition 3.4 (Mechanism). *We define a mechanism \mathcal{M} for a setting (n, \mathcal{PI}) as a tuple $\mathcal{M} := (\mathcal{R}, M)$ where \mathcal{R} is a set of all possible reports the mechanism allows, and $M : \mathcal{R}^n \mapsto \mathbb{R}^n$ is a mapping from all agents' reports to each agent's reward.*

The mechanism requires agents to submit a report r . For example, r can simply be an agent's private information. In this case, $\mathcal{R} = \mathcal{PI}$. We define \mathbf{r} to be a report profile (r_1, r_2, \dots, r_n) where r_i is agent i 's report.

Definition 3.5 (Strategy). *Given a mechanism \mathcal{M} , we define the strategy of \mathcal{M} for setting (n, \mathcal{PI}) as a mapping s from (PI, Q) (private signal and prior received) to a probability distribution over \mathcal{R} .*

We define a strategy profile \mathbf{s} as a profile of all agents' strategies $\{s_1, s_2, \dots, s_n\}$ and we say agents play \mathbf{s} if for any i , agent i plays strategy s_i . Let \mathcal{S} be the set of all strategy profiles.

Note that actually the definition of a strategy profile only depends on the setting and the definition of all possible reports \mathcal{R} . We will need the definition of a mechanism when we define an equilibrium.

3.1.1 Special Strategy Profiles

For different settings, with different \mathcal{R} , there can be different definitions for the "truth-telling" strategy. We say a definition for a truth-telling strategy \mathbf{T} is a **permissible truth-telling** if for every unknown common prior there is fixed mapping to translate the report profile \mathbf{r} to every agent's private information when everyone plays \mathbf{T} . For example, in the case agents are asked to report their private information directly $\mathcal{R} = \mathcal{PI}$, a permissible definition for the truth-telling strategy is the strategy where an agent always reports her private information truthfully. By abusing notation a little, we still use \mathbf{T} to represent the strategy profile $(\mathbf{T}, \mathbf{T}, \dots, \mathbf{T})$ where everyone plays \mathbf{T} .

We will see when the mechanism knows no information about the prior profile, no non-trivial mechanism has truth-telling as the *unique* "best" equilibrium. Thus, it is too much to ask for a strictly focal or strictly dominant focal mechanism with respect to all non-truthful equilibria. The

best we can hope is to construct a mechanism which is strictly focal or strictly dominant focal with respect to all non-truthful equilibria excluding all *permutation strategy profiles* (Definition 3.8) when the mechanism only knows the prior is symmetric; or all non-truthful equilibria excluding all *generalized permutation strategy profiles* (Definition 3.9). In the later sections, we will construct the mechanisms which are in this sense, optimal. Because permutation strategies seem unnatural, risky, and require the same amount of effort as truth-telling these are still strong guarantees.

A permutation $\pi : \mathcal{PI} \mapsto \mathcal{PI}$ can be seen as a relabelling of private information. Given two lists of permutations $\pi = (\pi_1, \pi_2, \dots, \pi_n)$, $\pi' = (\pi'_1, \pi'_2, \dots, \pi'_n)$, we define the product of π and π' as

$$\pi \cdot \pi' := (\pi_1 \cdot \pi'_1, \pi_2 \cdot \pi'_2, \dots, \pi_n \cdot \pi'_n)$$

where for every i , $\pi_i \cdot \pi'_i$ is the group product of π_i and π'_i such that $\pi_i \cdot \pi'_i$ is a new permutation with $\pi_i \cdot \pi'_i(PI) = \pi_i(\pi'_i(PI))$ for any PI .

We also define π^{-1} as $(\pi_1^{-1}, \pi_2^{-1}, \dots, \pi_n^{-1})$.

By abusing notation a little, we define $\pi : \mathcal{Q} \mapsto \mathcal{Q}$ as a mapping from a prior Q to a *generalized permuted prior* $\pi(Q)$ where for any $PI_1, PI_2, \dots, PI_n \in \mathcal{PI}$,

$$Pr_{\pi(Q)}(PI_1, PI_2, \dots, PI_n) = Pr_Q(\pi_1^{-1}(PI_1), \pi_2^{-1}(PI_2), \dots, \pi_n^{-1}(PI_n))$$

where PI_i is the private signal of agent i . Notice that it follows that:

$$Pr_{\pi(Q)}(\pi_1(PI_1), \pi_2(PI_2), \dots, \pi_n(PI_n)) = Pr_Q(PI_1, PI_2, \dots, PI_n).$$

Intuitively, $\pi(Q)$ is the same as Q after the signals are relabelled according to π .

For every agent i , given her strategy is s_i , we define $\pi(s_i)$ as the strategy such that $\pi(s_i)(PI, Q) = s_i(\pi_i(PI), \pi(Q))$ for every private information PI and prior Q .

For convenience, we write $(\pi, \pi, \dots, \pi)(Q)$ as $\pi(Q)$ and $(\pi, \pi, \dots, \pi)(s)$ as $\pi(s)$.

Definition 3.6 (Generalized Permuted Strategy Profile). *For any strategy profile \mathbf{s} , we define $\pi(\mathbf{s})$ as a strategy profile with $\pi(\mathbf{s}) = (\pi(s_1), \pi(s_2), \dots, \pi(s_n))$.*

Note that $\pi^{-1}\pi Q = Q$ which implies $\pi^{-1}\pi(\mathbf{s}) = \mathbf{s}$.

We define a permutation strategy (profile) and then give a generalized version of this definition.

Definition 3.7 (Permutation Strategy). *We define a strategy s as a permutation strategy if there exists a permutation π such that $s = \pi(\mathbf{T})$.*

Definition 3.8 (Permutation Strategy Profile). *We define a strategy profile \mathbf{s} as a permutation strategy profile if there exists a permutation π such that $\mathbf{s} = \pi(\mathbf{T})$.*

Definition 3.9 (Generalized Permutation Strategy Profile). *We define a strategy profile \mathbf{s} as a generalized permutation strategy profile if there exists a list of permutations $\pi = (\pi_1, \pi_2, \dots, \pi_n)$ such that $\mathbf{s} = \pi(\mathbf{T}) = (\pi_1, \pi_2, \dots, \pi_n)(\mathbf{T})$.*

3.2 Mechanism Design Goals

In this section, we give a criterion for a mechanism to be “good”. We first introduce the definition of an information monotone function (IMF) which we will use as a building block when we define a “good mechanism”.

From agent i 's prospective, before she receives her private information, the report profile $\mathbf{r} = (r_1, r_2, \dots, r_n)$ are random variables. The distribution of \mathbf{r} agent i believes depends both on the prior Q_i agent i has and the strategy profiles \mathbf{s} agents play, thus, we sometimes write \mathbf{r} as $\mathbf{r}(Q_i, \mathbf{s})$.

An IMF takes the report profile as input. For every agent i , the expectation of an IMF is maximized when all agents play a truth-telling strategy profile \mathbf{T} . We can think of an IMF as way to measure the amount of information contained in agents' reports. Intuitively, when everyone uses a permissible truth-telling strategy, the information contained in agents' reports is maximized. Formally, we define IMF in the below definition:

Definition 3.10 (Information Monotone Function). *Given setting (n, \mathcal{PI}) , a set of possible reports \mathcal{R} , truth-telling strategy \mathbf{T} , prior profile \mathbf{Q} and a set of strategy profiles \mathcal{S}^* , we say a function*

$$IMF : \mathcal{R}^n \mapsto \mathbb{R}$$

is an information monotone function for $(n, \mathcal{PI}, \mathcal{R}, \mathbf{T}, \mathbf{Q}, \mathcal{S}^)$ if from any agent i prospective (given her prior Q_i), for any strategy profile $\mathbf{s} \in \mathcal{S}^*$, the expectation of $IMF(\mathbf{r})$ when agents play \mathbf{s} is less than the expectation of $IMF(\mathbf{r})$ when agents play \mathbf{T} , which we write as*

$$\mathbb{E}_{\mathbf{r}(Q_i, \mathbf{s})}[IMF(\mathbf{r})] \leq \mathbb{E}_{\mathbf{r}(Q_i, \mathbf{T})}[IMF(\mathbf{r})]$$

Moreover, we say IMF is strictly information monotone if the equality holds iff $\mathbf{s} = \mathbf{T}$.

Now we define a stronger version of the information monotone function definition—a pairwise information monotone function (PIMF) which is useful when designing “good” mechanisms. The only difference between IMF and PIMF is that PIMF only takes two agents' reports as input rather than all agents' reports.

Definition 3.11 (Pairwise Information Monotone Function). *Given a setting (n, \mathcal{PI}) , a set of possible reports \mathcal{R} , a truth-telling strategy \mathbf{T} , prior profile \mathbf{Q} and a set of strategy profiles \mathcal{S}^* , we say a function*

$$PIMF : \mathcal{R}^2 \mapsto \mathbb{R}$$

is a pairwise information monotone function for $(n, \mathcal{PI}, \mathcal{R}, \mathbf{T}, \mathbf{Q}, \mathcal{S}^)$ if from any agent i 's prospective (given her prior Q_i), for any strategy profile $\mathbf{s} \in \mathcal{S}^*$, for any two agents j, k , the expectation of $PIMF(r_j, r_k)$ when agents play \mathbf{s} is less than the expectation of $PIMF(r_j, r_k)$ when agents play \mathbf{T} , which we write as*

$$\mathbb{E}_{(r_j, r_k)(Q_i, (s_j, s_k))}[PIMF(r_j, r_k)] \leq \mathbb{E}_{(r_j, r_k)(Q_i, \mathbf{T})}[PIMF(r_j, r_k)]$$

where (r_j, r_k) is agent j, k 's reports which are random variables that depend on Q_i and agent j, k 's strategies.

Moreover, we say PIMF is strictly information monotone if the equality holds iff $\mathbf{s} = \mathbf{T}$.

Definition 3.12 (Agent Welfare). *Given a mechanism \mathcal{M} , for a strategy profile \mathbf{s} , we define the agent welfare of \mathbf{s} as the sum of expected payments to agents when they play \mathbf{s} under \mathcal{M} .*

Mechanism Design Goals Given a permissible truth-telling strategy, and a set of strategy profiles \mathcal{S}^* ,

1. we say a mechanism is truthful if it always has truth-telling as an equilibrium;
2. we say a mechanism is (strictly) dominant focal with respect to \mathcal{S}^* if it is truthful and each agent's expected payment is an (strict) information monotone function for \mathcal{S}^* ;
3. we say a mechanism is (strictly) focal with respect to \mathcal{S}^* if it is truthful and the agent welfare of this mechanism is an (strict) information monotone function for \mathcal{S}^* .

Given a setting and a truthful mechanism \mathcal{M} , recall that \mathcal{S} as all possible strategy profiles agents can play. We also define $\mathcal{S}_{\mathcal{M}}$ as all possible equilibrium of \mathcal{M} .

Ideally, we hope $\mathcal{S}^* = \mathcal{S}$. However, it is usually good enough if a mechanism is dominant focal (or focal) with respect to the set of its equilibria $\mathcal{S}_{\mathcal{M}}$ since non-equilibrium strategy profiles seem to be unstable. For strictness guarantee, we will see in Section 9, it is impossible to have a strictly dominant focal (or strictly focal) mechanism even just with respect to all equilibria when the mechanism knows little information about the prior.

3.3 Mechanism Design via PIMF (IMF)

In this section, we will show the relationship between PIMFs (IMFs) and mechanism design goals. The relationship is stated in the below informal theorem:

Theorem 3.15 (Informal). *(I) For any setting (n, \mathcal{PT}) and a report set \mathcal{R} , if we can find a PIMF for all possible strategy profiles, we can use this PIMF to design a mechanism that is dominant focal with respect to all possible strategy profiles.*

(II) Given a setting and a truthful mechanism \mathcal{M} , if we can find a PIMF for all possible equilibria of \mathcal{M} , we can use this PIMF to modify \mathcal{M} to $\mathcal{M}+$ where $\mathcal{M}+$ is focal with respect to all its equilibria.

Definition 3.13 (\mathcal{M}_{PIMF}). *Given a setting (n, \mathcal{PT}) , a set \mathcal{R} , and a PIMF for \mathcal{S} , we define \mathcal{M}_{PIMF} as the mechanism in which*

1. *Each agent i is paired with a randomly chosen reference agent j .*
2. *Each agent i is paid $PIMF(r_i, r_j)$ where r_j is agent j 's report.*

Shortly, we will formally prove that \mathcal{M}_{PIMF} satisfies statement (I) in Theorem 3.15.

To modify a mechanism \mathcal{M} to $\mathcal{M}+$ via PIMF: we (a) first use a typical trick to create a zero-sum game which has the same equilibria as the previous mechanism \mathcal{M} ; and (b) then pay each agent an extra score that only depends on other agents which will not change the structure of the equilibria. We want this extra score to be determined by a PIMF.

Definition 3.14 (Modified Mechanism $\mathcal{M}+(PIMF)$). *Given a setting, a truthful mechanism \mathcal{M} , and a PIMF for $\mathcal{S}_{\mathcal{M}}$, we modify \mathcal{M} to $\mathcal{M}+(PIMF)$ with the below steps¹:*

1. *(Zero-Sum Trick) Divide the agents into two non-empty groups—group A and group B . Each group of agents plays the mechanism \mathcal{M} restricted to their own group. For group A , each agent i_A receives a*

$$score_{\mathcal{M}}(i_A, \mathbf{r}) = payment_{\mathcal{M}}(i_A, \mathbf{r}_A) - \frac{1}{|A|} \sum_{j_B \in B} payment_{\mathcal{M}}(j_B, \mathbf{r}_B)$$

where $payment_{\mathcal{M}}(i_A, \mathbf{r}_A)$ is agent i_A 's payment under the restricted mechanism \mathcal{M} given group A ' reports \mathbf{r}_A and $payment_{\mathcal{M}}(j_B, \mathbf{r}_B)$ is agent j_B 's payment under restricted mechanism \mathcal{M} given group B ' reports \mathbf{r}_B . We score agents in group B analogously.

2. *(Additional Information Measurement) Each agent i is paired with two random agents $j, k \in [n]$, $j, k \neq i$, the payment for agent i is*

$$payment_{\mathcal{M}+(PIMF)}(i, \mathbf{r}) = score_{\mathcal{M}}(i, \mathbf{r}) + PIMF(r_j, r_k)$$

¹This modification method is similar with the modification method in [9] (see Section 2).

Theorem 3.15 (Formal). (I) \mathcal{M}_{PIMF} is dominant focal with respect to \mathcal{S} .

(II) The set of all equilibria of $\mathcal{M}+(PIMF)$ is $\mathcal{S}_{\mathcal{M}}$ and $\mathcal{M}+(PIMF)$ is focal with respect to $\mathcal{S}_{\mathcal{M}}$. Moreover, the agent welfare in $\mathcal{M}+(PIMF)$ is proportional to the expectation of $\sum_{i,j} PIMF(r_i, r_j)$.

Proof. To show \mathcal{M}_{PIMF} is dominant focal with respect to \mathcal{S} , we need to show \mathcal{M}_{PIMF} is truthful and each agent's expected payment is an information monotone function for \mathcal{S} . For every agent i , she will be paid $PIMF(r_i, r_j)$ in \mathcal{M}_{PIMF} where agent j is her reference agent. If everyone else tells the truth, agent i 's expected payment will be reduced by playing a non-truthful strategy according to the definition of $PIMF$ (see definition 3.11). Thus, \mathcal{M}_{PIMF} is truthful. If agents play a non-truthful strategy profile, still, the expectation of $PIMF(r_i, r_j)$ will decrease according to the definition of $PIMF$. Therefore \mathcal{M}_{PIMF} is dominant focal.

To show $\mathcal{M}+(PIMF)$ is (quasi) focal, we first show (i) $\mathcal{M}+(PIMF)$ has the same equilibria as \mathcal{M} , we then show (ii) the agent welfare of $\mathcal{M}+(PIMF)$ is proportional to the expectation of $\sum_{j,k} PIMF(r_j, r_k)$. According to Definition 3.11, we know that the expectation $\sum_{j,k} PIMF(r_j, r_k)$ is maximized by truthful equilibrium. Combining (i), (ii) and the definition of $PIMF$, we can see $\mathcal{M}+(PIMF)$ is focal with respect to all its equilibria.

Proof of (i): $PIMF(r_j, r_k)$ does not depend on agent i 's strategy, and the part of $score_{\mathcal{M}}$ related to agent i 's strategy is contained in $payment_{\mathcal{M}}(i, \mathbf{r})$. This implies that agent i 's marginal benefit from deviation in $\mathcal{M}+(PIMF)$ is the same with its marginal benefit from the same deviation in \mathcal{M} . So $\mathcal{M}+(PIMF)$ has the same equilibria as \mathcal{M} .

Proof of (ii): If we sum over all agents' payments, the sum will be

$$\begin{aligned} & \sum_i \sum_{j,k \neq i} Pr(j,k) PIMF(r_j, r_k) \\ &= \sum_{j,k} (n-2) Pr(j,k) PIMF(r_j, r_k) \\ &= (n-2) \sum_{j,k} Pr(j,k) PIMF(r_j, r_k) \end{aligned}$$

Where $Pr(j,k)$ is the probability agent j, k are chosen. Thus, (ii) follows. \square

3.4 Information Monotonicity

Now the question is how to design PIMFs, we are going to use a tool from information theory—information monotonicity which was first used in peer-prediction literature by Kong and Schoenebeck [9].

We first introduce f -divergence—a measure for the difference between two probability distributions:

F -divergence [1] f -divergence $D_f : \Delta_{\Sigma} \times \Delta_{\Sigma} \rightarrow \mathbb{R}$ is a non-symmetric measure of difference between distribution $\mathbf{p} \in \Delta_{\Sigma}$ and distribution $\mathbf{q} \in \Delta_{\Sigma}$ and is defined to be

$$D_f(\mathbf{p}, \mathbf{q}) = \sum_{\sigma \in \Sigma} \mathbf{p}(\sigma) f\left(\frac{\mathbf{p}(\sigma)}{\mathbf{q}(\sigma)}\right)$$

where $f(\cdot)$ is a convex function. Now we introduce the properties of f -divergence:

1. **Non-negative:** For any \mathbf{p}, \mathbf{q} , $D_f(\mathbf{p}, \mathbf{q}) \geq 0$ and $D_f(\mathbf{p}, \mathbf{q}) = 0$ if and only if $\mathbf{p} = \mathbf{q}$.
2. **Information Monotonicity:** For any \mathbf{p}, \mathbf{q} , and transition mapping $\theta : \Sigma \times \Sigma \mapsto [0, 1]$ satisfying $\forall \sigma \in \Sigma, \sum_{\sigma'} \theta(\sigma, \sigma') = 1$, we have $D_f(\mathbf{p}, \mathbf{q}) \geq D_f(\theta\mathbf{p}, \theta\mathbf{q})$.

Lemma 3.16 (Information Monotonicity ([1])). *For any strictly convex function f , f -divergence $D_f(\mathbf{p}, \mathbf{q})$ satisfies information monotonicity so that for any transition matrix $\theta \in \mathbb{R}^{\Sigma \times \Sigma}$, $D_f(\mathbf{p}, \mathbf{q}) \geq D_f(\theta\mathbf{p}, \theta\mathbf{q})$.*

Moreover, the inequality is strict if and only if there exists $\sigma, \sigma', \sigma''$ such that $\theta(\sigma, \sigma')\mathbf{p}(\sigma') > 0$, $\theta(\sigma, \sigma'')\mathbf{p}(\sigma'') > 0$ and $\frac{\mathbf{q}(\sigma')}{\mathbf{p}(\sigma')} \neq \frac{\mathbf{q}(\sigma'')}{\mathbf{p}(\sigma')}$.

We put the proof to appendix for reference.

Definition 3.17. *Given two signals $\sigma', \sigma'' \in \Sigma$, we say two probability measures \mathbf{p}, \mathbf{q} over Σ can **distinguish** $\sigma', \sigma'' \in \Sigma$ if $\mathbf{p}(\sigma') > 0$, $\mathbf{p}(\sigma'') > 0$ and $\frac{\mathbf{q}(\sigma')}{\mathbf{p}(\sigma')} \neq \frac{\mathbf{q}(\sigma'')}{\mathbf{p}(\sigma')}$.*

Lemma 3.16 directly implies

Corollary 3.18. *Given a transition matrix θ and two probability measures \mathbf{p}, \mathbf{q} that can distinguish $\sigma', \sigma'' \in \Sigma$, if there exists $\sigma \in \Sigma$ such that $\theta(\sigma, \sigma'), \theta(\sigma, \sigma'') > 0$, we have $D_f(\mathbf{p}, \mathbf{q}) > D_f(\theta\mathbf{p}, \theta\mathbf{q})$ where f is a strictly convex function.*

Now we introduce two f -divergences in common use: KL divergence and Toal Variance Distance.

Example 3.19 (KL divergence). *If we choose the convex function $f(x)$ as $\log(x)$, then we obtain KL divergence $D_{KL}(\mathbf{p}, \mathbf{q}) = \sum_{\sigma} \mathbf{p}(\sigma) \log \frac{\mathbf{p}(\sigma)}{\mathbf{q}(\sigma)}$.*

Example 3.20 (Toal Variance Distance). *If we pick convex function $f(x)$ as $|x - 1|$, we can obtain Toal Variance Distance $D_{tvd}(\mathbf{p}, \mathbf{q}) = \sum_{\sigma} |\mathbf{p}(\sigma) - \mathbf{q}(\sigma)|$.*

4 The Multiple Questions Setting

In this section, we introduce the multiple questions setting which was previously studied in Dasgupta and Ghosh [2] and Radanovic and Faltings [14]: n agents are assigned the same m questions. For each question k , each agent i receives a **private signal** $\sigma_i^k \in \Sigma$ about question k and is asked to report this signal. In this setting, $\mathcal{PI} = \Sigma^m$. However, agent i may lie and report $\hat{\sigma}_i^k \neq \sigma_i^k$. Dasgupta and Ghosh [2] give the following example for this setting: n workers are asked to check the quality of m goods, they may receive signal “high quality” or “low quality”. In particular, they assume $\Sigma = \{0, 1\}$.

Agents have priors for questions. Each agent i believes agents’ private signals for question k are chosen from a joint distribution Q_i^k over Σ^n . Note that different agents may have different priors for the same question and each agent may have different priors for different questions.

In the multiple questions setting, people usually make the below assumption:

Assumption 4.1 (A Priori Similar and Random Order). *For any i , any $k \neq k'$, $Q_i^k = Q_i^{k'}$. Moreover, all questions appear in a random order, independently drawn for each agent.*

This means agents cannot distinguish each question without the private signal they receive.

Each agent i ’s **strategy** for each question k is defined as a mapping $\theta_i^k : \Sigma \times \Sigma \mapsto [0, 1]$ where $\sum_{\hat{\sigma}_i^k} \theta_i^k(\sigma_i^k, \hat{\sigma}_i^k) = 1$ for any σ_i^k and $\theta_i^k(\sigma_i^k, \hat{\sigma}_i^k)$ is the probability agent i reports $\hat{\sigma}_i^k$ conditioned on receiving σ_i^k . We say agent i plays a **consistent strategy** if for any k, k' , $\theta_i^k = \theta_i^{k'}$.

We define **truth-telling \mathbf{T}** as the strategy where an agent truthfully reports her private signal for every question. Note that \mathbf{T} is a consistent strategy. We define a **strategy profile** $\theta = (\theta_1, \theta_2, \dots, \theta_n)$ as a profile of strategies where for every i , agent i plays strategy θ_i . By abusing notation a little, we still use \mathbf{T} to represent the strategy profile where everyone tells the truth (plays \mathbf{T}). We define a **consistent strategy profile** as the strategy profile where all agents play a consistent strategy.

With the a priori similar and random order assumption, Dasgupta and Ghosh [2] make the below observation:

Observation 4.2. [2] *When questions are a priori similar and agents receive questions in random order (Assumption 4.1), agents can only use a consistent strategy.*

5 The f -Mutual Information Mechanism

In this section, we first give the definition of f -mutual information which is a generalization version of the traditional definition of mutual information in Section 5.1. We will show f -mutual information is maximized by the truthful strategy profile. Section 5.2 describes f -mutual information mechanism that can be applied to the non-binary setting and shows f -mutual information mechanism is (strictly) dominant focal. In Section 5.3 shows Dasgupta and Ghosh [2] mechanism is a special case of f -mutual information mechanism (in the binary setting) with the f -mutual information using the specific f -divergence, total variation distance.

5.1 f -Mutual Information

In this section, we are going to introduce a concept that is similar to mutual information. This concept will help us design information monotone functions for the multiple questions setting.

Definition 5.1. *For two agents i, j , let $(\hat{\sigma}_i, \hat{\sigma}_j)$ be the pair of agent i and agent j 's reported signals for a randomly chosen question. Note that the distribution of $(\hat{\sigma}_i, \hat{\sigma}_j)$ depends on agent i and agent j 's strategies.*

Let $U_{ij}^{\mathbf{s}}$ and $V_{ij}^{\mathbf{s}}$ be two probability measures over $\Sigma \times \Sigma$ where $U_{ij}^{\mathbf{s}}$ is the joint distribution of $(\hat{\sigma}_i, \hat{\sigma}_j)$ and $V_{ij}^{\mathbf{s}}$ is the product of the marginal distributions of $\hat{\sigma}_i$ and $\hat{\sigma}_j$, given agents play strategy \mathbf{s} . Formally, for every $(\hat{\sigma}_i, \hat{\sigma}_j) \in \Sigma \times \Sigma$,

$$U_{ij}^{\mathbf{s}}((\hat{\sigma}_i, \hat{\sigma}_j)) = \Pr[(\hat{\sigma}_i, \hat{\sigma}_j)]$$

and

$$V_{ij}^{\mathbf{s}}((\hat{\sigma}_i, \hat{\sigma}_j)) = \Pr[\hat{\sigma}_i] \Pr[\hat{\sigma}_j]$$

Now we introduce a definition that is a general version of the normal definition of mutual information where we use f -divergence instead of KL divergence.

Definition 5.2 (f -Mutual Information). *Given agents play strategy profile \mathbf{s} , we define the f -mutual information $f\text{-}MI_{ij}^{\mathbf{s}}$ between agent i and agent j as*

$$f\text{-}MI_{ij}^{\mathbf{s}} = D_f(U_{ij}^{\mathbf{s}}, V_{ij}^{\mathbf{s}})$$

When both agent i and agent j tell the truth, $U_{ij}^{\mathbf{s}}$ and $V_{ij}^{\mathbf{s}}$ become $U_{ij}^{\mathbf{T}}$ and $V_{ij}^{\mathbf{T}}$. Let $f\text{-}MI_{ij}^{\mathbf{T}} = D_f(U_{ij}^{\mathbf{T}}, V_{ij}^{\mathbf{T}})$ be the f -mutual information between agent i and agent j when both agent i and agent j tell the truth.

We say a $f\text{-}MI$ is strictly convex if f is a strictly convex function.

For strictness guarantee, we introduce the following assumption:

Assumption 5.3 (Fine-grained Prior). *In the multiple questions setting, for every agent i , we say her prior is fine-grained if for every agent j , for every two distinct pairs $(\sigma', \sigma''), (\tilde{\sigma}', \tilde{\sigma}'')$, $U_{ij}^{\mathbf{T}}$ and $V_{ij}^{\mathbf{T}}$ can **distinguish** (see Definition 3.17) (σ', σ'') and $(\tilde{\sigma}', \tilde{\sigma}'')$.*

Lemma 5.4. *For any consistent strategy profile \mathbf{s} , we have*

$$f\text{-}MI_{ij}^{\mathbf{s}} \leq f\text{-}MI_{ij}^{\mathbf{T}}$$

for any two agents i, j .

Moreover, if agent i 's prior is fine-grained and $f\text{-}MI$ is strictly convex, agent i will believe $f\text{-}MI_{ij}^{\mathbf{s}} < f\text{-}MI_{ij}^{\mathbf{T}}$ when one of θ_i, θ_j is not a permutation where θ_i, θ_j are agent i, j 's consistent strategies respectively.

Note that the result of this lemma is valid in a very general setting: the number of possible signals can be any integer and there is no restriction on the priors.

Proof for Lemma 5.4. For every two consistent strategies θ_i, θ_j , we will construct a transition mapping Θ that maps both $U_{ij}^{\mathbf{s}}$ and $V_{ij}^{\mathbf{s}}$ to $U_{ij}^{\mathbf{T}}$ and $V_{ij}^{\mathbf{T}}$. We then apply information monotonicity to show the result:

$$f\text{-}MI_{ij}^{\mathbf{s}} = D_f(U_{ij}^{\mathbf{s}}, V_{ij}^{\mathbf{s}}) \tag{1}$$

$$= D_f(\Theta U_{ij}^{\mathbf{T}}, \Theta V_{ij}^{\mathbf{T}}) \tag{2}$$

$$\leq D_f(U_{ij}^{\mathbf{T}}, V_{ij}^{\mathbf{T}}) \tag{3}$$

$$= f\text{-}MI_{ij}^{\mathbf{T}} \tag{4}$$

It remains to construct Θ . For every two consistent strategies θ, θ' , we define a new mapping $\theta \otimes \theta' : \Sigma^2 \times \Sigma^2 \mapsto [0, 1]$ where

$$\theta \otimes \theta'((\sigma, \sigma'), (\hat{\sigma}, \hat{\sigma}')) = \theta(\sigma, \hat{\sigma}) \times \theta'(\sigma', \hat{\sigma}')$$

Given strategy profile \mathbf{s} we construct Θ' as a $\Sigma^2 \times \Sigma^2$ matrix representation of $\theta_i \otimes \theta_j$ such that the $((\sigma_i, \sigma_j), (\hat{\sigma}_i, \hat{\sigma}_j))$ entry is $\theta_i \otimes \theta_j((\sigma_i, \sigma_j), (\hat{\sigma}_i, \hat{\sigma}_j))$.

We define Θ as the transpose of Θ' . Note that $\sum_{(\hat{\sigma}_i, \hat{\sigma}_j)} \Theta((\hat{\sigma}_i, \hat{\sigma}_j), (\sigma_i, \sigma_j)) = 1$ for any (σ_i, σ_j) which implies the sum of any column of Θ is 1. Thus Θ is a transition matrix.

By marginalization and conditional probability formula, we have

$$Pr(\hat{\sigma}_i, \hat{\sigma}_j) = \sum_{\sigma_i, \sigma_j} \theta_i(\sigma_i, \hat{\sigma}_j) \theta_j(\sigma_j, \hat{\sigma}_j) Pr(\sigma_i, \sigma_j)$$

and

$$Pr(\hat{\sigma}_i) = \sum_{\sigma_i} \theta_i(\sigma_i, \hat{\sigma}_i) Pr(\sigma_i), Pr(\hat{\sigma}_j) = \sum_{\sigma_j} \theta_j(\sigma_j, \hat{\sigma}_j) Pr(\sigma_j)$$

which implies

$$U_{ij}^{\mathbf{s}} = \Theta U_{ij}^{\mathbf{T}}, V_{ij}^{\mathbf{s}} = \Theta V_{ij}^{\mathbf{T}}$$

Therefore we have finished the construction of transition mapping Θ .

Now we show the strictness guarantee. When either agent i or j plays a non-permutation strategy, we can see $\theta_i \otimes \theta_j$ is not permutation. Thus there must exist $(x', x''), (\sigma', \sigma''), (\tilde{\sigma}', \tilde{\sigma}'') \in \Sigma \times \Sigma$ such that both $\theta_i \otimes \theta_j((x', x''), (\sigma', \sigma''))$ and $\theta_i \otimes \theta_j((x', x''), (\tilde{\sigma}', \tilde{\sigma}'')) > 0$ where $(\sigma', \sigma'') \neq (\tilde{\sigma}', \tilde{\sigma}'')$. According to the definition of fine-grained prior (see Definition 5.3), $U_{ij}^{\mathbf{T}}$ and $V_{ij}^{\mathbf{T}}$ can distinguish (σ', σ'') and $(\tilde{\sigma}', \tilde{\sigma}'')$. Thus, $f\text{-}MI_{ij}^{\mathbf{s}} < f\text{-}MI_{ij}^{\mathbf{T}}$ according to Corollary 3.18. \square

5.2 The f -Mutual Information Mechanism

If the number of questions is infinite, then we can calculate $U_{ij}^{\mathbf{s}}, V_{ij}^{\mathbf{s}}$ for any ij via a unbiased estimator. Note that according to the definition of $U_{ij}^{\mathbf{s}}, V_{ij}^{\mathbf{s}}, f\text{-}MI_{ij}^{\mathbf{s}}$, the calculation of $U_{ij}^{\mathbf{s}}, V_{ij}^{\mathbf{s}}, f\text{-}MI_{ij}^{\mathbf{s}}$ does not require any knowledge of \mathbf{s} .

Definition 5.5. We define the f -Mutual Information Mechanism $\mathcal{M}_{\mathcal{MI}}$ as follows:

1. Each agent i is paired with a randomly chosen reference agent j .
2. Each agent i is paid $f\text{-}MI_{ij}^{\mathbf{s}} = D_f(U_{ij}^{\mathbf{s}}, V_{ij}^{\mathbf{s}})$.

Theorem 5.6. $\mathcal{M}_{\mathcal{MI}}$ is dominant focal with respect to all consistent strategy profiles. Moreover, when every agent's prior is fine-grained, $\mathcal{M}_{\mathcal{MI}}$ is strictly dominant focal with respect to all consistent non-permutation strategy profiles.

Proof. We first observe that when agents play non-permutation strategy profile, then for each agent i , either her strategy θ_i is not a permutation or with positive probability, her reference agent, call him agent j , has strategy θ_j that is not a permutation.

With the above observation and Lemma 5.4, we can see the payment of $\mathcal{M}_{\mathcal{MI}}$ for each agent is (strict) information monotone function with respect to all consistent (non-permutation) strategy profiles.

Therefore, the theorem follows. \square

Aided by Observation 4.2, the below corollary follows:

Corollary 5.7. With the a priori similar and random order assumption (4.1), $\mathcal{M}_{\mathcal{MI}}$ is dominant focal with respect to all strategy profiles. Moreover, when every agent's prior is fine-grained, $\mathcal{M}_{\mathcal{MI}}$ is strictly dominant focal with respect to all consistent non-permutation strategy profiles.

The impossibility result in Section 9 will show that the strictly dominant focal guarantee for $\mathcal{M}_{\mathcal{MI}}$ is optimal.

5.3 Proof of Dasgupta and Ghosh [2013] in Our Framework

We first state the mechanism M_d and the main theorem in Dasgupta and Ghosh [2].

Mechanism M_d Agents are asked to report binary signals 0 or 1 for each question. Randomly pick a reference agent j for agent i . For each question k , pick subsets $A \subseteq [m] \setminus k, B \subseteq [m] \setminus (k \cup A)$ with $|A| = |B| = d$. We define $\bar{\sigma}_i^A = \frac{\sum_{l \in A} \hat{\sigma}_i^l}{|A|}$ to be agent i 's average answer for subset A , $\bar{\sigma}_j^B = \frac{\sum_{l \in B} \hat{\sigma}_j^l}{|B|}$ is agent j 's average answer for subset B .

Agent i 's reward for each question k is

$$R_{i,j}^k := [\hat{\sigma}_i^k * \hat{\sigma}_j^k + (1 - \hat{\sigma}_i^k) * (1 - \hat{\sigma}_j^k)] - [\bar{\sigma}_i^A * \bar{\sigma}_j^B + (1 - \bar{\sigma}_i^A) * (1 - \bar{\sigma}_j^B)]$$

Intuitively, M_d pays each agent according to the correlation between her answer and other agents' answers.

Dasgupta and Ghosh [2] also make an additional assumption:

Assumption 5.8 (Positively Correlated). *For every k , each question k has a unknown ground truth a^k and for every agent i , with probability greater or equal to $\frac{1}{2}$, agent i receives private signal a^k .*

Theorem 5.9. [2] *With the a priori similar questions assumption (see Assumption 4.1) and the positively correlated assumption (see Assumption 5.8), the case everyone tells the truth is a equilibrium. (ii) The expected reward of each agent when everyone tells the truth is at least as great as that in any other equilibrium.*

We note that a slightly stronger version of (i) is proved in [2]: truth-telling is a strict Bayesian Nash equilibrium.

Reproof Outline Note that in expectation, each agent is paid the correlation between her answers and her reference agent j 's answers. We will show the absolute value of correlation is a specific f -mutual information—total variation distance- MI . Then the result follows from the information monotone property of f -mutual information.

Note that the convex function $|x - 1|$ used to obtain total variance distance is not a strictly convex function. This is why we cannot apply the strictness part in Lemma 5.4.

Lemma 5.10 ($|\text{Correlation}| = \text{Total Variation Distance-}MI$). *For any two random **binary** variables $X, Y \in \{-1, 1\}$, we have*

$$|\text{Cov}(X, Y)| = \sum_{x, y} |Pr(X = x, Y = y) - Pr(X = x)Pr(Y = y)|$$

Proof. For convenience, we define $a_{xy} = Pr(X = x, Y = y)$, $b_{xy} = Pr(X = x)Pr(Y = y)$.

$$\text{Cov}(X, Y) = \mathbb{E}(X * Y) - \mathbb{E}(X)\mathbb{E}(Y) \tag{5}$$

$$= Pr(X = 1, Y = 1) + Pr(X = -1, Y = -1) \tag{6}$$

$$- Pr(X = 1, Y = -1) - Pr(X = -1, Y = 1) \tag{7}$$

$$- (Pr(X = 1) - Pr(X = -1))(Pr(Y = 1) - Pr(Y = -1)) \tag{8}$$

$$= (a_{1,1} - b_{1,1}) + (a_{-1,-1} - b_{-1,-1}) + (b_{1,-1} - a_{1,-1}) + (b_{-1,1} - a_{-1,1}) \tag{9}$$

Since $a_{1,1} + a_{-1,1} = b_{1,1} + b_{-1,1}$, we have $a_{1,1} \geq b_{1,1} \Leftrightarrow a_{-1,1} \leq b_{-1,1}$, similarly, we have $a_{1,1} \geq b_{1,1} \Leftrightarrow a_{1,-1} \leq b_{1,-1} \Leftrightarrow a_{-1,-1} \geq b_{-1,-1}$. This implies that the four parts in (9) always have the same sign. Thus, the sum of the absolute value is the absolute value of the sum. The lemma follows. \square

Proof of Theorem 5.9. By simple calculations, we can see when questions are a priori similar, prior to seeing question k , the expected reward of agent i for question k when paired with reference agent j is

$$2 * (\mathbb{E}(\hat{\sigma}_i^k * \hat{\sigma}_j^k) - \mathbb{E}(\hat{\sigma}_i^k) * \mathbb{E}(\hat{\sigma}_j^k))$$

Say p_i, p_j are the probability agent i, j get question k correctly respectively. The positively correlated assumption $p_i, p_j > \frac{1}{2}$ implies that for every question k ,

$$(\mathbb{E}(\sigma_i^k * \sigma_j^k)) - \mathbb{E}(\sigma_i^k) * \mathbb{E}(\sigma_j^k) \quad (10)$$

$$= Pr(a_k = 1) * p_i * p_j + Pr(a_k = 0) * (1 - p_i) * (1 - p_j) \quad (11)$$

$$- (Pr(a_k = 1) * p_i + Pr(a_k = 0) * (1 - p_i)) * (Pr(a_k = 1) * p_j + Pr(a_k = 0) * (1 - p_j)) \quad (12)$$

$$> 0 \quad (13)$$

which means when both agent i, j are truthful, their answers are positively correlated. Thus, the expected reward of agent i for any question is positive when all agents are truthful.

By Observation 4.2, we know agents only play consistent strategies. Now we start to show that for every question k , $|\mathbb{E}(\hat{\sigma}_i^k \hat{\sigma}_j^k) - \mathbb{E}(\hat{\sigma}_i^k)E(\hat{\sigma}_j^k)|$ is a pairwise information monotone function for the consistent strategy profiles, then we use this fact as a tool to simply prove the theorem.

Without loss of generality, we will omit the k .

First note that we can adjust $\hat{\sigma}_i, \hat{\sigma}_j$ to $\{-1, 1\}$ since $2 * [\mathbb{E}(\hat{\sigma}_i \hat{\sigma}_j) - \mathbb{E}(\hat{\sigma}_i)E(\hat{\sigma}_j)] = \mathbb{E}((2 * \hat{\sigma}_i - 1)(2 * \hat{\sigma}_j - 1)) - \mathbb{E}(2 * \hat{\sigma}_i - 1)E(2 * \hat{\sigma}_j - 1)$. By Lemma 5.10, $|\mathbb{E}(\hat{\sigma}_i \hat{\sigma}_j) - \mathbb{E}(\hat{\sigma}_i)E(\hat{\sigma}_j)|$ is a specific f -mutual information where f -divergence is chosen as total variation distance. Thus, $|\mathbb{E}(\hat{\sigma}_i \hat{\sigma}_j) - \mathbb{E}(\hat{\sigma}_i)E(\hat{\sigma}_j)|$ is a pairwise information monotone function for the consistent strategy profiles based on Lemma 5.4.

For each agent i who is paired with a randomly chosen agent (call him agent j), if everyone else tells the truth, she knows the randomly chosen agent j tells the truth as well. The expected reward she will receive is

$$\mathbb{E}(\hat{\sigma}_i \sigma_j) - \mathbb{E}(\hat{\sigma}_i)E(\sigma_j) \quad (14)$$

$$\leq |\mathbb{E}(\hat{\sigma}_i \sigma_j) - \mathbb{E}(\hat{\sigma}_i)E(\sigma_j)| \quad (15)$$

$$\leq |\mathbb{E}(\sigma_i \sigma_j) - \mathbb{E}(\sigma_i)E(\sigma_j)| \quad (16)$$

$$= \mathbb{E}(\sigma_i \sigma_j) - \mathbb{E}(\sigma_i)E(\sigma_j) \quad (17)$$

(16) follows since we have already proved that $|\mathbb{E}(\hat{\sigma}_i \hat{\sigma}_j) - \mathbb{E}(\hat{\sigma}_i)E(\hat{\sigma}_j)|$ is a pairwise information monotone function for the consistent strategy profiles (agent j plays truth-telling which is a consistent strategy profile as well).

(17) follows since $\mathbb{E}(\sigma_i \sigma_j) - \mathbb{E}(\sigma_i)E(\sigma_j)$ is always non-negative because of the positively correlated assumption.

For agent i , to maximize her expected reward, she should tell the truth as well. Therefore, the case that all agents tell the truth is an equilibrium.

Similarly, we can also use the information monotone property of total variation distance and positively correlated assumption to show that

$$\mathbb{E}(\hat{\sigma}_i \hat{\sigma}_j) - \mathbb{E}(\hat{\sigma}_i)E(\hat{\sigma}_j) \leq \mathbb{E}(\sigma_i \sigma_j) - \mathbb{E}(\sigma_i)E(\sigma_j)$$

where $\mathbb{E}(\hat{\sigma}_i \hat{\sigma}_j) - \mathbb{E}(\hat{\sigma}_i)E(\hat{\sigma}_j)$ is agent i 's expected reward conditioning on paired with agent j when all agents play consistent strategy profile.

That means, if agents play other consistent strategy profile rather than truth-telling, the expected reward of each agent i is less than that when everyone tells the truth. Therefore, Theorem 5.9 follows. \square

6 The One Question Setting with Common Prior

In this section, we will introduce a basic setting—the one question setting in the peer-prediction literature. We will also introduce the signal-prediction framework ([12]) and proper scoring rules

([18]) that are widely used in designing mechanisms in one question setting. We will show the application of our information monotone framework for the one question setting in Section 7 and Section 8. The mechanism we consider in Section 7 applies to a small group of agents but requires an additional self-predicting assumption (see Assumption 7.7) about the prior. The mechanism we consider in Section 8 only applies to a large group of agents but does not require the additional self-predicting assumption (see Assumption 7.7) about the prior.

The one question setting is the setting where agents are asked to answer one question. Each agent i receives a **private signal** $\sigma_i \in \Sigma$ for this question and is asked to report this signal. Note that in this setting $\mathcal{PI} = \Sigma$.

Agents have priors for the joint distribution of private signals that all agents receive.

Assumption 6.1 (Common and Symmetric Prior). *We assume agents share a common symmetric prior.*

That is, agents share the same prior and the inference each agent's signal lets her draw about others' signals does not depend on her identity or on the identity of the other agent.

With common and symmetric prior assumptions, if agents receive the same private signal, they have the same posterior for the world.

Signal-prediction Framework Now we introduce the signal-prediction framework of the mechanism proposed in [12]. In this mechanism, each agent i is not only required to report her private signal σ_i but also a prediction $p_i \in \Delta_\Sigma$ (recall that Δ_Σ is the set of all possible probability measures over Σ) for other agents' signals. In the signal-prediction framework, \mathcal{R} becomes $\Sigma \times \Delta_\Sigma$. Her payment depends on two scores she receives: the **prediction score** and the **information score**. At a high level, the **prediction score** measures how good the agent's prediction is, the **information score** incentivizes agents to truthfully report their received signals. Usually, we use proper scoring rules which we will introduce to measure the **prediction score**. The **information score** can be customized.

Now we introduce the definition of strategy in the mechanism that is based on the signal-prediction framework. For each agent i , we define $p_i := \mathbf{q}_{\sigma_i}$ as a probability measure where for any $\sigma \in \Sigma$, $\mathbf{q}_{\sigma_i}(\sigma)$ is agent i 's expectation for the fraction of other agents who have received σ given she received σ_i . According to the common prior and the symmetric prior assumption, $\mathbf{q}_{\sigma_i}(\sigma) = \Pr_Q(\sigma_j = \sigma | \sigma_i)$ for every j where Q is the common prior. Agent i may report $\hat{p}_i \neq \mathbf{q}_{\sigma_i}$. We define $r_i := \{\hat{\sigma}_i, \hat{p}_i\}$ as a report agent i reports.

For convenience, when we consider the strategy each agent i uses, we fix the prior she has and only consider the mapping s_i from PI to a probability distribution over \mathcal{R} . In the signal-prediction framework, the **strategy** of each agent i is a mapping $s_i : \Sigma \mapsto \Delta_{\mathcal{R}}$ from a received private signal σ_i to a distribution over the reports of the signal-prediction pair. We write the marginal distribution of s_i for agent i 's signal report as a mapping $\theta_i : \Sigma \times \Sigma \mapsto [0, 1]$ with $\sum_{\hat{\sigma}_i} \theta_i(\sigma_i, \hat{\sigma}_i) = 1$ for any $\sigma_i \in \Sigma$. We call θ_i the **signal strategy** of s_i . We also call $(\theta_1, \theta_2, \dots, \theta_n)$ the **signal strategy profile** of (s_1, s_2, \dots, s_n) .

We define a strategy profile $\mathbf{s} := (s_1, s_2, \dots, s_n)$ as a profile of all agents' strategies and we say agents play \mathbf{s} when each agent i plays s_i .

For every agent i , we say she uses **truth-telling** strategy \mathbf{T} if she always reports $(\sigma_i, \mathbf{q}_{\sigma_i})$ given she receives σ_i . By abusing notation a little, we still use \mathbf{T} to represent the strategy profile where everyone tells the truth (plays \mathbf{T}).

Proper Scoring Rules (PSR) We introduce strictly proper scoring rules [18] here. Starting with [11], proper scoring rules have become a common ingredient in mechanisms for unverifiable

information elicitation (e.g. [12, 20]).

A scoring rule $PS : \Sigma \times \Delta_\Sigma \rightarrow \mathbb{R}$ takes in a signal $\sigma \in \Sigma$ and a distribution over signals $\delta_\Sigma \in \Delta_\Sigma$ and outputs a real number. A scoring rule is *proper* if, whenever the first input is drawn from a distribution δ_Σ , then the expectation of PS is maximized by δ_Σ . A scoring rule is called *strictly proper* if this maximum is unique. We will assume throughout that the scoring rules we use are strictly proper. By slightly abusing notation, we can extend a scoring rule to be $PS : \Delta_\Sigma \times \Delta_\Sigma \rightarrow \mathbb{R}$ by simply taking $PS(\delta_\Sigma, \delta'_\Sigma) = \mathbb{E}_{\sigma \leftarrow \delta_\Sigma}(PS(\sigma, \delta'_\Sigma))$. We note that this means that any proper scoring rule is linear in the first term.

Example 6.2 (Example of Proper Scoring Rule). *Fix a finite outcome space Σ for a signal σ . Let $\mathbf{q} \in \Delta_\Sigma$ be a reported distribution. The Logarithmic Scoring Rule maps a signal and reported distribution to a payoff as follows:*

$$L(\sigma, \mathbf{q}) = \log(\mathbf{q}(\sigma)).$$

Let the signal σ be drawn from some random process with distribution $\mathbf{p} \in \Delta_\Sigma$. Then the expected payoff of the Logarithmic Scoring Rule

$$\mathbb{E}_{\sigma \leftarrow \mathbf{p}}[L(\sigma, \mathbf{q})] = \sum_{\sigma} \mathbf{p}(\sigma) \log \mathbf{q}(\sigma) = L(\mathbf{p}, \mathbf{q})$$

According to Winkler [18], this value will be maximized if and only if $\mathbf{q} = \mathbf{p}$.

Prediction Score via PSR

1. Each agent i is paired with a randomly chosen reference agent $j \neq i$.
2. Each agent i receives the prediction score $PS(\hat{\sigma}_j, \hat{p}_i)$.

Recall that for every agent i , $\mathbf{q}_{\sigma_i}(\sigma)$ is her expectation for the fraction of agents who have *received* σ . It only depends on the prior and agent i 's private signal σ_i . However, agent i 's expectation for the fraction of agents who have *reported* σ depends on the prior, σ_i and the strategy profile \mathbf{s} agents play.

Definition 6.3. *For every agent i , we define $\mathbf{q}_{\sigma_i}^{\mathbf{s}}$ as the probability measure where for any $\hat{\sigma} \in \Sigma$, $\mathbf{q}_{\sigma_i}^{\mathbf{s}}(\hat{\sigma}) = \Pr(\hat{\sigma} | \sigma_i, \mathbf{s})$ is agent i 's expectation for the fraction of agents (excluding agent i) who has reported $\hat{\sigma}$.*

It follows from the properties of Proper Scoring Rules that to maximize the prediction score, each agent i should report her belief for the random variable $\hat{\sigma}_j$ which depends on agent i 's prior, private signal and the strategies other agents use. Thus, the below claim follows.

Claim 6.4. *For each agent i , $\mathbf{q}_{\sigma_i}^{\mathbf{s}}$ is the unique best response for agent i to maximize her prediction score when other agents play \mathbf{s}_{-i} .*

7 f -Disagreement and Decomposable Payment Schemes

In this section, we will design a focal mechanism for a small group of agents in the one question setting. This focal mechanism is constructed by modifying a truthful decomposable payment scheme: Multi-valued RBTS proposed by Radanovic and Faltings [13]. In Section 7.1, we will introduce the main tool— f -disagreement we will use in modifying Multi-valued RBTS. In Section 7.2, we will introduce the related prior work about decomposable payment scheme and Multi-valued RBTS by Radanovic and Faltings [13]. In Section 7.3, we will construct the focal mechanism which can apply a small group of agents.

7.1 Prior Work: f -Disagreement

In this section, we introduce a concept called f -Disagreement that is essentially the same as the *Diversity* concept used in Kong and Schoenebeck [9].

Definition 7.1 (f -Disagreement [9]). *Define the f -disagreement $f-DI_{ij}^{\mathbf{s}}$ between two agents i, j as $D_f(\mathbf{q}_{\sigma_i}^{\mathbf{s}}, \mathbf{q}_{\sigma_j}^{\mathbf{s}})$ when agents play \mathbf{s} . When everyone tells the truth, $f-DI_{ij}^{\mathbf{s}}$ becomes $f-DI_{ij}^{\mathbf{T}}$.*

Kong and Schoenebeck [9] shows that $f-DI_{ij}^{\mathbf{s}} \leq f-DI_{ij}^{\mathbf{T}}$ with some conditions (See Lemma 7.5). To give the strictness guarantee of $f-DI_{ij}^{\mathbf{s}} \leq f-DI_{ij}^{\mathbf{T}}$, Kong and Schoenebeck [9] introduced three additional assumptions:

Assumption 7.2 (Non-zero Prior). *We assume that for any $\sigma, \sigma' \in \Sigma$, $q(\sigma) > 0, q(\sigma|\sigma') > 0$.*

Assumption 7.3 (Informative Prior). *We assume if agents have different private signals, they will have different expectations for the fraction of at least one signal. That is for any $\sigma \neq \sigma'$, there exists σ'' such that $q(\sigma''|\sigma) \neq q(\sigma''|\sigma')$.*

Assumption 7.4 (Fine-grained Prior). *In one question with common prior setting, we assume that for any $\sigma \neq \sigma' \in \Sigma$, there exists σ'', σ''' such that*

$$\frac{q(\sigma|\sigma'')}{q(\sigma'|\sigma'')} \neq \frac{q(\sigma|\sigma''')}{q(\sigma'|\sigma''')}$$

Lemma 7.5 ([9]). *When the common prior is non-zero, informative, and fine-grained (i) for any symmetric strategy profile \mathbf{s} , $\sum_{ij} f-DI_{ij}^{\mathbf{s}} \leq \sum_{ij} f-DI_{ij}^{\mathbf{T}}$. (ii) When the number of agents is infinite, for any strategy profile \mathbf{s} , $\sum_{ij} f-DI_{ij}^{\mathbf{s}} \leq \sum_{ij} f-DI_{ij}^{\mathbf{T}}$.*

The inequality is strict when the signal strategy profile of \mathbf{s} is not a permutation strategy profile.

7.2 Prior Work: Decomposable Payment Scheme

In this section, we introduce the related work in Radanovic and Faltings [13].

Definition 7.6 (Decomposable Payment Scheme (DPS)). [13] *A decomposable payment scheme (mechanism) is a payment scheme based on signal-prediction framework with the additional requirement: each agent's information score does not depend on the prediction she reported.*

This main result requires the following assumption in addition to the common and symmetric prior assumption:

Assumption 7.7 (Self-Predicting [13]). *For any two signals $\sigma \neq \sigma'$,*

$$Pr(\sigma|\sigma') < Pr(\sigma|\sigma)$$

This means, for any agent i , her expectation for the fraction of agents who receive σ is maximized when she also receives σ .

With this additional assumption, they design a truthful mechanism for non-binary signals and small group of agents:

Multi-valued RBTS [13] This mechanism follows the signal-prediction framework. The prediction score is measured by a proper scoring rule. For the information score, each agent i is paired with a reference agent $j = i + 1(\text{mod } n)$ and receives an information score: $\frac{1}{\bar{p}_j(\hat{\sigma}_i)} \mathbb{1}_{\hat{\sigma}_i = \hat{\sigma}_j}$

Theorem 7.8. [13] *With the common prior, symmetric prior, self-predicting prior assumptions, Multi-valued RBTS is strictly truthful.*

7.3 Constructing Focal Mechanisms for a Small Group of Agents via f -Disagreement

For any decomposable payment scheme \mathcal{DPS} , we will design a PIMF that modifies \mathcal{DPS} to $\mathcal{DPS}+(PIMF)$ where the agent welfare of truth-telling is higher than that in any other symmetric equilibrium and if the number of agents n goes to infinity, the expected total payment of agents when everyone tells the truth is higher than that in any other equilibrium.

Definition 7.9. We define $f-DI : \mathcal{R}^2 \mapsto \mathbb{R}$ as a function such that for any two reports $r_j = (\hat{\sigma}_j, \hat{p}_j), r_k = (\hat{\sigma}_k, \hat{p}_k)$, $f-DI(r_j, r_k) = D_f(\hat{p}_j, \hat{p}_k)$.

Note that this definition is very similar to our f -disagreement except that \hat{p}_j and \hat{p}_k may not be $\mathbf{q}_{\sigma_j}^s$ and $\mathbf{q}_{\sigma_k}^s$. However, we will show if the mechanism is a DPS, then if \mathbf{s} is an equilibrium, each agent i will report $\mathbf{q}_{\sigma_i}^s$ given that her private signal is σ_i .

Modified Decomposable Payment Scheme $\mathcal{DPS}+(f-DI)$ We modify a \mathcal{DPS} to $\mathcal{DPS}+(f-DI)$ via Definition 3.14.

Theorem 7.10. When agents share a common and symmetric prior, if \mathcal{DPS} is truthful, $\mathcal{DPS}+(f-DI)$ is focal with respect to (i) all symmetric equilibria; (ii) all equilibria when the number of agents is infinite.

Moreover, with the additional non-zero, informative, fine-grained prior assumptions, $\mathcal{DPS}+(f-DI)$ is strictly focal with respect to (i) all symmetric non-permutation equilibria; (ii) all non-permutation equilibria when the number of agents is infinite.

Proof for Theorem 7.10. According to Theorem 3.15, we only need to show $f-DI$ is a pairwise information monotone function for (a) symmetric equilibrium; (b) any equilibrium when the number of agents is infinite.

Note that $\mathcal{DPS}+(f-DI)$ has the same equilibrium as \mathcal{DPS} . The definition of \mathcal{DPS} tells us each agent's information score does not depend on her prediction. Thus, for each agent i , the prediction score is the only part in her payment that depends on her prediction. When she maximizes her payment in equilibrium \mathbf{s} , she will report $\mathbf{q}_{\sigma_i}^s$ to maximize her prediction score according to Claim 6.4.

Based on Lemma 7.5, $f-DI$ is a pairwise information monotonicity function for (a) symmetric equilibrium; (b) any equilibrium when the number of agents is infinite. Therefore, for mechanism $\mathcal{DPS}+(f-DI)$, given any common prior Q , the agent welfare of truth-telling is higher than any symmetric equilibrium; if the number of agents is infinite, the agent welfare of truth-telling is higher than any equilibrium.

When the common prior is non-zero, informative and fine-grained, based on Lemma 7.5, the function that maps \mathbf{r} to $\sum_{i,j} f-DI(r_i, r_j)$ is strictly information monotone for (a) all symmetric non-permutation equilibria; (b) all non-permutation equilibria when the number of agents is infinite. According to Theorem 3.15, $\mathcal{DPS}+(f-DI)$ is strictly focal with respect to (a) symmetric non-permutation equilibria; (b) all non-permutation equilibria when the number of agents is infinite. \square

Combining Theorem 7.8 and Theorem 7.10, the below corollary directly follows.

Corollary 7.11. When agents share a common symmetric and self-predicting prior, Multi-valued $\mathcal{RBTS}+(f-DI)$ is focal with respect to (i) all symmetric equilibria; (ii) all equilibria when the number of agents is infinite.

Moreover, with the additional non-zero, informative, fine-grained prior assumptions, Multi-valued $\mathcal{RBTS}+(f-DI)$ is strictly focal with respect to (i) all symmetric non-permutation equilibria; (ii) all non-permutation equilibria when the number of agents is infinite.

Multi-valued RBTS+(f -DI) does not know the prior but knows the fact that the prior is symmetric. Thus the impossibility result in Kong and Schoenebeck [9] shows that the strictly focal guarantee for Multi-valued RBTS+(f -DI) is optimal.

Moreover, we note the result of this corollary is weaker than Kong and Schoenebeck [9]’s results since Corollary 7.11 requires self-predicting assumption (see Assumption 7.7) for truthfulness. However, the analysis of this mechanism follows easily from our framework, were as the analysis that Kong and Schoenebeck [9]’s Disagreement mechanism is extraordinarily involved.

8 f -Information Gain and Bayesian Truth Serum

In this section, we first give the definition of f -information gain which is a general version of the traditional definition of information gain (relative entropy) in Section 8.1. We will show f -information gain is maximized by the truth-telling strategy profile. In Section 8.2, we introduce the prior related work of Prelec [12]. We then use the information monotone property of f -information gain to reprove Prelec [12]’s main results in Section 8.3.

8.1 f -Information Gain

Definition 8.1. For every two agents i, j , let $\hat{\sigma}_{-ij}$ be the list of reported signals $\hat{\sigma}$ excluding $\hat{\sigma}_i$ and $\hat{\sigma}_j$. Note that $\hat{\sigma}_{-ij} \in \Sigma^{n-2}$.

Let $P_{\sigma_i}^{\mathbf{s}}$ and $P_{\sigma_i, \sigma_j}^{\mathbf{s}}$ be two probability measures defined so that for every $\hat{\sigma}_{-ij} \in \Sigma^{n-2}$,

$$\begin{aligned} P_{\sigma_i}^{\mathbf{s}}(\hat{\sigma}_{-ij}) &= \Pr(\hat{\sigma}_{-ij} | \sigma_i) \\ P_{\sigma_i, \sigma_j}^{\mathbf{s}}(\hat{\sigma}_{-ij}) &= \Pr(\hat{\sigma}_{-ij} | \sigma_i, \sigma_j) \end{aligned}$$

for every $x \in \Sigma^{n-2}$, where $\hat{\sigma}$ is the list of reported signals when agents play strategy \mathbf{s} . Thus, for agents i, j , $P_{\sigma_i}^{\mathbf{s}}$ is the prediction for other agents’ reported signals conditioning on agent i ’s private signal σ_i and the fact that agents play strategy profile \mathbf{s} ; $P_{\sigma_i, \sigma_j}^{\mathbf{s}}$ is the prediction for other agents’ reported signals conditioning on both agent i and agent j ’s private signals σ_i, σ_j and the fact that agents play strategy profile \mathbf{s} .

When everyone tells the truth: $\hat{\sigma}_{-ij}$ becomes σ_{-ij} ; $P_{\sigma_i}^{\mathbf{s}}$ becomes $P_{\sigma_i}^{\mathbf{T}}$ and $P_{\sigma_i, \sigma_j}^{\mathbf{s}}$ becomes $P_{\sigma_i, \sigma_j}^{\mathbf{T}}$.

Definition 8.2 (f -Information Gain). Given that agents play strategy profile \mathbf{s} , we define agent i ’s f -information gain $f\text{-IG}_{\sigma_i, \sigma_j}^{\mathbf{s}}$ from agent j as

$$f\text{-IG}_{\sigma_i, \sigma_j}^{\mathbf{s}} = D_f(P_{\sigma_i, \sigma_j}^{\mathbf{s}}, P_{\sigma_i}^{\mathbf{s}})$$

Let $f\text{-IG}_{\sigma_i, \sigma_j}^{\mathbf{T}} = D_f(P_{\sigma_i, \sigma_j}^{\mathbf{T}}, P_{\sigma_i}^{\mathbf{T}})$ be agent i ’s information gain from agent j when everyone tells the truth.

Lemma 8.3. For any strategy profile \mathbf{s} , $f\text{-IG}_{\sigma_i, \sigma_j}^{\mathbf{s}} \leq f\text{-IG}_{\sigma_i, \sigma_j}^{\mathbf{T}}$.

Proof. To prove this lemma, we will show there exists a transition mapping Θ from $\Sigma^{n-2} \times \Sigma^{n-2}$ to $[0, 1]$ such that $P_{\sigma_i}^{\mathbf{s}} = \Theta P_{\sigma_i}^{\mathbf{T}}$, $P_{\sigma_i, \sigma_j}^{\mathbf{s}} = \Theta P_{\sigma_i, \sigma_j}^{\mathbf{T}}$. Once we have Θ , the lemma follows from information monotonicity:

$$f\text{-IG}_{\sigma_i, \sigma_j}^{\mathbf{s}} = D_f(P_{\sigma_i, \sigma_j}^{\mathbf{s}}, P_{\sigma_i}^{\mathbf{s}}) = D_f(\Theta P_{\sigma_i, \sigma_j}^{\mathbf{T}}, \Theta P_{\sigma_i}^{\mathbf{T}}) \leq f\text{-IG}_{\sigma_i, \sigma_j}^{\mathbf{T}}$$

It remains to construct Θ :

For any $\hat{\sigma}_{-ij} \in \Sigma^{n-2}$,

$$P_{\sigma_i}^s(\hat{\sigma}_{-ij}) = Pr(\hat{\sigma}_{-ij}|\sigma_i) \quad (18)$$

$$= \sum_{\sigma_{-ij} \in \Sigma^{n-2}} (\Pi_{k \neq i,j} \theta_k((\sigma_{-ij})_k, (\hat{\sigma}_{-ij})_k)) Pr(\sigma_{-ij}|\sigma_i) \quad (19)$$

$$= \sum_{\sigma_{-ij} \in \Sigma^{n-2}} (\Pi_{k \neq i,j} \theta_k((\sigma_{-ij})_k, (\hat{\sigma}_{-ij})_k)) P_{\sigma_i}^T(\sigma_{-ij}) \quad (20)$$

where θ_k is the signal strategy of agent k .

Note that $Pr(\sigma_{-ij}|\sigma_i)$ is the probability that agents excluding i, j receive private signals σ_{-ij} . Since each agent plays their strategy independently, $(\Pi_{k \neq i,j} \theta_k((\sigma_{-ij})_k, (\hat{\sigma}_{-ij})_k))$ is the probability that agents excluding i, j reports $\hat{\sigma}_{-ij}$, conditioning agents excluding i, j receive private signals σ_{-ij} respectively. Thus (19) follows from Bayes' rule.

We define $\bigotimes_{k \neq i,j} \theta_k$ as a mapping from $\Sigma^{n-2} \times \Sigma^{n-2}$ to $[0, 1]$ such that

$$\bigotimes_{k \neq i,j} \theta_k(\sigma_{-ij}, \hat{\sigma}_{-ij}) = \Pi_{k \neq i,j} \theta_k((\sigma_{-ij})_k, (\hat{\sigma}_{-ij})_k)$$

where θ_k is the signal strategy of agent k when agents play \mathbf{s} .

Given strategy profile \mathbf{s} we construct Θ' as a $\Sigma^{n-2} \times \Sigma^{n-2}$ matrix representation of $\bigotimes_{k \neq i,j} \theta_k$ such that the $(\sigma_{-ij}, \hat{\sigma}_{-ij})$ entry is $\bigotimes_{k \neq i,j} \theta_k(\sigma_{-ij}, \hat{\sigma}_{-ij})$.

We define Θ as the transpose of Θ' . Note that $\sum_{\hat{\sigma}_{-ij}} \Theta(\hat{\sigma}_{-ij}, \sigma_{-ij}) = 1$ for any σ_{-ij} which implies the sum of any column of Θ is 1. Thus Θ is a transition mapping. Moreover, (20) can be written as

$$P_{\sigma_i}^s = \Theta P_{\sigma_i}^T, P_{\sigma_i, \sigma_j}^s = \Theta P_{\sigma_i, \sigma_j}^T$$

Therefore we have finished the construction of transition mapping Θ . \square

8.2 Prior Work: Bayesian Truth Serum

In this section, we are going to introduce the related work in Prelec [12]. Prelec [12] proposed Bayesian Truth Serum in one question setting. In addition to the common prior and the symmetric prior assumptions, two additional assumptions are required:

Assumption 8.4 (Conditional Independence). *There exists a unknown probability measure ω over Σ and according to ω , agents' private signals are independently and identically distributed. That is, for every i , agent i receives signal σ with probability $\omega(\sigma)$.*

Assumption 8.5 (Large Group). *The number of agents is infinite.*

BTS derives a prior distribution using an aggregate of these forecasts, and then rewards agents for giving signal reports that are “unexpectedly common” with respect to this distribution. Intuitively, an agent will believe her private signal is underestimated by other agents which means she will believe the actual fraction of her own private signal is higher than the average of agents' predictions.

Mechanism Bayesian Truth Serum (BTS(α)) [12] Each agent i has two scores: a **prediction score** and an **information score**. BTS pays each agent

$$\text{prediction score} + \alpha \cdot \text{information score}$$

where $\alpha > 1$

To calculate the scores, for every agent i , the mechanism randomly chooses a reference agent $j \neq i$, agent i 's prediction score is

$$\text{score}_{Pre}(r_i, r_j) := L(\hat{\sigma}_j, \hat{p}_i) = \log \hat{p}_i(\hat{\sigma}_j)$$

where \hat{p}_i is agent i 's reported prediction and $\hat{\sigma}_j$ is agent j 's reported signal. Recall that $L(\cdot, \cdot)$ is the log scoring rule. Thus the prediction score measures the accuracy of agent i 's prediction. Agent i 's information score is

$$\text{score}_{Im}(r_i, r_j) := \log \frac{fr(\hat{\sigma}_i | \hat{\sigma}_{-i})}{\hat{p}_j(\hat{\sigma}_i)}$$

where $fr(\hat{\sigma}_i | \hat{\sigma}_{-i})$ is the fraction of all reported signals $\hat{\sigma}_{-i}$ (excluding agent i) that agree with agent i 's reported signal $\hat{\sigma}_i$. Intuitively, the signals that actually occur more than agents believe they will receive a higher information score.

Now we restate the main theorem about Bayesian Truth Serum:

Theorem 8.6. [12] *With the common prior, the symmetric prior, the conditional independence, and the large group assumptions, in BTS(α), truth-telling is a Bayesian Nash equilibrium, the expected information score of each agent when everyone tells the truth is higher than that in any other equilibrium. Moreover, when $\alpha > 1$, BTS(α) is focal with respect to all its equilibria.*

To show theorem 8.6, Prelec [12] uses the below claim:

Claim 8.7. *In BTS(α), when $\alpha > 1$, the agent welfare is proportional to the sum of all agents' information scores.*

and shows a derivation that we can succinctly interpret as the following lemma which gives the relationship between information score and information gain:

Lemma 8.8. [12] *Given agents play equilibrium \mathbf{s} , for every agent i ,*

$$\mathbb{E}(\text{score}_{Im}(r_i, r_j) | j, \mathbf{s}) \leq KL\text{-}IG_{\sigma_j, \sigma_i}^{\mathbf{s}}$$

where $\mathbb{E}(\text{score}_{Im}(r_i, r_j) | j, \mathbf{s})$ is the expected information score of agent i conditioning on her reference agent being agent j , and $KL\text{-}IG$ is the f -information gain when f -divergence is chosen as KL divergence.

The equality holds when everyone tells the truth.

With the above claim and lemma, Prelec [12] uses some clever algebraic calculations to prove the main results.

We put Prelec [12]'s proof for Lemma 8.8 to the appendix.

8.3 Using Our Framework to Analyse BTS

Lemma 8.3 shows that the information gain is maximized by truthful strategy profile. Lemma 8.8 gives the relation between information score and information gain. Theorem 8.6 follows almost immediately from Lemma 8.3 and Lemma 8.8 in our framework.

Proof of Theorem 8.6. We first show that each agent's information score is maximized when everyone tells the truth.

With the results of Lemma 8.8 and Lemma 8.3, we have

$$\mathbb{E}(\text{score}_{Im}(r_i, r_j)|j, \mathbf{s}) \leq KL\text{-}IG_{\sigma_j, \sigma_i}^{\mathbf{s}} \leq KL\text{-}IG_{\sigma_j, \sigma_i}^{\mathbf{T}} = \mathbb{E}(\text{score}_{Im}(r_i, r_j)|j, \mathbf{T}) \quad (21)$$

The first inequality follows from Lemma 8.8. The second inequality follows from Lemma 8.3.

Lemma 8.8 also shows that when everyone tells the truth, the expected information score $\mathbb{E}(\text{score}_{Im}(r_i, r_j)|j, \mathbf{T})$ equals $KL\text{-}IG_{\sigma_j, \sigma_i}^{\mathbf{T}}$.

To see that truth-telling is a Bayesian Nash equilibrium, note that according to Equation 21 for each agent i , her information score is maximized when everyone (including herself) tells the truth. Thus, truth-telling is an equilibrium since for each agent i , when everyone else tells the truth, she can maximize her information score by telling the truth as well. Equation 21 also immediately implies that the expected information score in other equilibrium is less than that in truth-telling equilibrium.

It is left to show $\text{BTS}(\alpha)$ is focal when $\alpha > 1$. Actually combining Claim 8.7 and the result we just proved, it immediately follows that $\text{BTS}(\alpha)$ is focal when $\alpha > 1$. □

9 Impossibility Results

In this section, we will show an impossibility result implies the optimality of the f -mutual information mechanism (Section 5.2). Recall that this mechanism was strictly dominant focal for all equilibria excluding generalized permutations strategy profiles. The results of this section imply that no mechanism can be strictly focal with respect to generalized permutations strategy profiles. We note that the optimality of the Multivalued RBTS+ (f -DI) mechanism designed in Section 7.3 follows from a similar result in Kong and Schoenebeck [9].

Definition 9.1. *Given a prior profile $\mathbf{Q} = (Q_1, Q_2, \dots, Q_n)$ and a strategy profile $\mathbf{s} = (s_1, s_2, \dots, s_n)$, and a mechanism \mathcal{M} , for every agent i , we define*

$$\nu_i^{\mathcal{M}}(n, \mathcal{PI}, \mathbf{Q}, \mathbf{s})$$

as agent i 's ex ante expected payment when agents play \mathbf{s} and all agents' private information are drawn from Q_i that is, from agent i 's viewpoint.

The impossibility result is stated as below:

Proposition 9.2. *Let \mathcal{M} be a mechanism that does not know the prior profile, then for any strategy profile s , and any permutation list π :*

- (1) *\mathbf{s} is a strict Bayesian Nash equilibrium of \mathcal{M} for any prior profile iff $\pi(\mathbf{s})$ is a strict Bayesian Nash equilibrium of \mathcal{M} for any prior profile.*
- (2) *For every agent i , there exists a prior profile \mathbf{Q} such that $\nu_i^{\mathcal{M}}(n, \mathcal{PI}, \mathbf{Q}, \mathbf{s}) \leq \nu_i^{\mathcal{M}}(n, \mathcal{PI}, \mathbf{Q}, \pi(\mathbf{s}))$.*

The proof of this theorem is very similar with the proof of the impossibility results in Kong and Schoenebeck [9]. We defer this proof to appendix.

Proposition 9.2 implies

Corollary 9.3. *Let \mathcal{M} be a mechanism, given truth-telling strategy \mathbf{T} , when \mathcal{M} knows no information about the prior profile of agents, if there exists a permutation list π such that $\pi(\mathbf{T}) \neq \mathbf{T}$, \mathbf{T} cannot be always paid strictly higher than all generalized permutation strategy profiles.*

We note that the requirement that there exists π such that $\pi(\mathbf{T}) \neq \mathbf{T}$ only fails for very trivial mechanisms where the truthfully reported strategy does not depend on the signal an agent receives.

The result in Kong and Schoenebeck [9] applies to the case when mechanism knows that agents share a symmetric common prior. Otherwise the result is the same as Corollary 9.3 except the guarantee only applies to permutation strategy profiles rather than generalized permutation strategy profiles.

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10 Appendix

Proposition 9.2. *Let \mathcal{M} be a mechanism that does not know the prior profile, for any strategy profile s , and any permutation π :*

- (1) *s is a strict Bayesian Nash equilibrium of \mathcal{M} for any prior profile iff $\pi(s)$ is a strict Bayesian Nash equilibrium of \mathcal{M} for any prior profile.*
- (2) *For every agent i , there exists a prior profile \mathbf{Q} such that $\nu_i^{\mathcal{M}}(n, \mathcal{PI}, \mathbf{Q}, s) \leq \nu_i^{\mathcal{M}}(n, \mathcal{PI}, \mathbf{Q}, \pi(s))$.*

The key idea to prove this theorem is what we refer to as **Indistinguishable Scenarios**:

Definition 10.1 (Scenario). *We define a scenario for the setting (n, \mathcal{PI}) as a tuple (\mathbf{Q}, s) where \mathbf{Q} is a prior profile, and s is a strategy profile.*

Given mechanism \mathcal{M} , for any scenario $A = (\mathbf{Q}_A, s_A)$, we write $\nu_{i_A}^{\mathcal{M}}(n, \mathcal{PI}, A)$ as agent i_A 's ex ante expected payment when agents play s_A and all agents' private signals are drawn from Q_{i_A} .

For two scenarios $A = (\mathbf{Q}_A, s_A)$, $B = (\mathbf{Q}_B, s_B)$ for setting (n, \mathcal{PI}) , let $PI_A := (PI_{1_A}, PI_{2_A}, \dots, PI_{n_A})$ be agents $(1_A, 2_A, \dots, n_A)$ ' private signals respectively in scenario A , $PI_B := (PI_{1_B}, PI_{2_B}, \dots, PI_{n_B})$ be agents $(1_B, 2_B, \dots, n_B)$ ' private signals respectively in scenario B .

Definition 10.2 (Indistinguishable Scenarios). *We say two scenarios A, B are indistinguishable $A \approx B$ if there is a coupling of the random variables PI_A and PI_B such that $\forall i, s_{i_A}(PI_{i_A}, Q_{i_A}) = s_{i_B}(PI_{i_B}, Q_{i_B})$ and agent i_A has the same belief about the world as agent i_B , in other words, for every j , $Pr(\hat{r}_{j_A} = \hat{r} | PI_{i_A}, \mathbf{Q}_A, s_A) = Pr(\hat{r}_{j_B} = \hat{r} | PI_{i_B}, \mathbf{Q}_B, s_B) \forall \hat{r} \in \mathcal{R}$.*

Now we will prove two properties of indistinguishable scenarios which are the main tools in the proof for our impossibility result.

Observation 10.3. *If $(\mathbf{Q}_A, s_A) \approx (\mathbf{Q}_B, s_B)$, then (i) for any mechanism \mathcal{M} , s_A is a (strict) equilibrium for the prior profile \mathbf{Q}_A iff s_B is a (strict) equilibrium for the prior profile \mathbf{Q}_B . (ii) $\forall i, \nu_{i_A}^{\mathcal{M}}(n, \mathcal{PI}, A) = \nu_{i_B}^{\mathcal{M}}(n, \mathcal{PI}, B)$*

At a high level, (1) is true since any reported profile distribution that agent i_A can deviate to, agent i_B can deviate to the same reported profile distribution as well and obtain the same expected payment as agent i_A .

Formally, we will prove the \Rightarrow direction in (1) by contradiction. The proof of the other direction will be similar. Consider the coupling for PI_A, PI_B mentioned in the definition of indistinguishable scenarios. For the sake of contradiction, assume there exists i and PI_{i_B} such that $\hat{r}' \neq s_{i_B}(PI_{i_B}, Q_{i_B})$ is a best response for agent i_B . Since agent i_A has the same belief about the world as agent i_B and $s_{i_A}(PI_{i_A}, Q_{i_A}) = s_{i_B}(PI_{i_B}, Q_{i_B})$, $\hat{r}' \neq s_{i_A}(PI_{i_A}, Q_{i_A})$ is a best response to agent i_A as well, which is a contradiction to the fact that s_A is a strictly equilibrium for prior Q_{i_A} .

To gain intuition about (2), consider the coupling again. For any i , agent i_A reports the same thing and has the same belief for the world as agent i_B , which implies the expected payoff of agent i_A is the same as agent i_B . (2) follows.

Now we are ready to prove our impossibility result:

of Proposition 9.2. We prove part (1) and part (2) separately.

Proof of Part (1) Let $A := (\mathbf{Q}, s), B := (\pi^{-1}(\mathbf{Q}), \pi(s))$. We will show that for any strategy profile s and any prior Q , $A \approx B$. Based on our above observations, part (1) immediately follows from that fact.

To prove $(Q, \mathbf{s}) \approx (\pi^{-1}Q, \pi(\mathbf{s}))$, for every i , we can couple $(PI_1, PI_2, \dots, PI_n)$ with $(\pi_1^{-1}(PI_1), \pi_2^{-1}(PI_2), \dots, \pi_n^{-1}(PI_n))$ where $(PI_1, PI_2, \dots, PI_n)$ is drawn from Q_i . It is a legal coupling since

$$Pr_{\pi^{-1}(Q_i)}(\pi_1^{-1}(PI_1), \pi_2^{-1}(PI_2), \dots, \pi_n^{-1}(PI_n)) = Pr_{Q_i}(PI_1, PI_2, \dots, PI_n)$$

according to the definition of $\pi^{-1}(Q)$.

Now we show this coupling satisfies the condition in Definition 10.2. First note that $\pi(s_i)(\pi^{-1}(PI_i), \pi^{-1}(Q)) = s_i(PI_i, Q)$. Now we begin to calculate $Pr(\hat{r}_{j_B} = \hat{r} | PI_{i_B}, \mathbf{Q}_B, \mathbf{s}_B)$

$$Pr(\hat{r}_{j_B} = \hat{r} | PI_{i_B}, \mathbf{Q}_B, \mathbf{s}_B) = Pr(\hat{r}_{j_B} = \hat{r} | \pi_i^{-1}(PI_{i_A}), \pi^{-1}(Q_{j_A}), \pi(\mathbf{s}_A)) \quad (22)$$

$$= \sum_{PI'} Pr_{\pi^{-1}(Q_{i_A})}(PI' | \pi_i^{-1}(PI_{i_A})) Pr(\pi(s_{j_A})(PI', \pi^{-1}(Q_{j_A})) = \hat{r}) \quad (23)$$

$$= \sum_{PI'} Pr_{\pi^{-1}(Q_{i_A})}(PI' | \pi_i^{-1}(PI_{i_A})) Pr(s_{j_A}(\pi(PI'), \pi\pi^{-1}(Q_{j_A})) = \hat{r}) \quad (24)$$

$$= \sum_{PI'} Pr_{Q_{i_A}}(\pi_j(PI') | PI_{i_A}) Pr(s_{j_A}(\pi(PI'), Q_{j_A}) = \hat{r}) \quad (25)$$

$$= \sum_{PI''} Pr_{Q_{i_A}}(PI'' | PI_{i_A}) Pr(s_{j_A}(PI'', Q_{i_A}) = \hat{r}) \quad (26)$$

$$= Pr(\hat{r}_{j_A} = \hat{r} | PI_{i_A}, \mathbf{Q}_A, \mathbf{s}_A) \quad (27)$$

From (22) to (23): To calculate the probability that agent j_B has reported \hat{r} , we should sum over all possible private signals agent j_B has received and calculate the probability agent j_B reported \hat{r} conditioning on he received private signal PI' , which is determined by agent j_B 's strategy $\pi(s_{j_A})$.

By abusing notation a little bit, we can write $\pi(s_{j_A})(PI', \pi^{-1}Q_{j_A})$ as a random variable (it is actually a distribution) with $Pr(\pi(s_{j_A})(PI', \pi^{-1}Q) = \hat{r}) = \pi(s_{j_A})(PI', \pi^{-1}Q)(\hat{r})$. According to above explanation, (23) follows.

(24) follows from the definition of permuted strategy (See Section 9).

(25) follows from the definition of permuted prior (See Section 9).

By replacing $\pi_j(PI')$ by PI'' , (26) follows.

We finished the proof $A \approx B$, as previously argued, result (1) follows.

Proof for Part (2) We will prove the second part by contradiction:

Fix permutation strategy profile π . First notice that there exists an positive integer O_d such that $\pi^{O_d} = I$ where I is the identity and agents play I means they tell the truth.

Given any strategy profile s , for the sake of contradiction, we assume that there exists a mechanism \mathcal{M} with unknown prior profile such that $\nu_{i_A}^{\mathcal{M}}(n, \mathcal{PI}, \mathbf{Q}, \mathbf{s}) > \nu_{i_A}^{\mathcal{M}}(n, \mathcal{PI}, \mathbf{Q}, \pi(\mathbf{s}))$ for any prior Q . For positive integer $k \in \{0, 1, \dots, O_d\}$, we construct three scenarios:

$$A_k := (\pi^k(\mathbf{Q}), \mathbf{s}), \quad A_{k+1} := (\pi^{k+1}(\mathbf{Q}), \mathbf{s}), \quad B_k := (\pi^k(\mathbf{Q}), \pi(\mathbf{s}))$$

and show for any k ,

$$(I) \nu_{i_A}^{\mathcal{M}}(n, \mathcal{PI}, A_k) > \nu_{i_A}^{\mathcal{M}}(n, \mathcal{PI}, B_k),$$

$$(II) \nu_{i_A}^{\mathcal{M}}(n, \mathcal{PI}, A_{k+1}) = \nu_{i_A}^{\mathcal{M}}(n, \mathcal{PI}, B_k).$$

Combining (I), (II) and the fact $A_0 = A_{O_d}$, we have

$$\nu_{i_A}^{\mathcal{M}}(n, \mathcal{PI}, A_0) > \nu_{i_A}^{\mathcal{M}}(n, \mathcal{PI}, A_1) > \dots \nu_{i_A}^{\mathcal{M}}(n, \mathcal{PI}, A_{O_d}) = \nu_{i_A}^{\mathcal{M}}(n, \mathcal{PI}, A_0)$$

which is a contradiction.

Now it is only left to show (I) and (II). Based on our assumption

$$\nu_{i_A}^M(n, \mathcal{PI}, \mathbf{Q}, \mathbf{s}) > \nu_{i_A}^M(n, \mathcal{PI}, \mathbf{Q}, \boldsymbol{\pi}(\mathbf{s}))$$

for any prior \mathbf{Q} , we have (I). By the same proof we have in part (1), we have $A_{k+1} \approx B_k$, which implies (II) according to our above observations. \square

Lemma 3.16 (Information Monotonicity ([1])). *For any strictly convex function f , f -divergence $D_f(\mathbf{p}, \mathbf{q})$ satisfies information monotonicity so that for any transition matrix $\theta \in \mathbb{R}^{\Sigma \times \Sigma}$, $D_f(\mathbf{p}, \mathbf{q}) \geq D_f(\theta\mathbf{p}, \theta\mathbf{q})$.*

Moreover, the inequality is strict if and only if there exists $\sigma, \sigma', \sigma''$ such that $\frac{\mathbf{p}(\sigma'')}{\mathbf{p}(\sigma')} \neq \frac{\mathbf{q}(\sigma'')}{\mathbf{q}(\sigma')}$ and $\theta(\sigma, \sigma')\mathbf{p}(\sigma') > 0$, $\theta(\sigma, \sigma'')\mathbf{p}(\sigma'') > 0$.

If the strictness condition does not satisfied, we can see $\theta\mathbf{p}$ and $\theta\mathbf{q}$ are \mathbf{p} and \mathbf{q} 's sufficient statistic which means the transition θ does not lose any information, thus, the equality holds.

Proof. The proof follows from algebraic manipulation and one application of convexity.

$$D_f(\theta\mathbf{p}, \theta\mathbf{q}) = \sum_{\sigma} (\theta\mathbf{p})(\sigma) f\left(\frac{(\theta\mathbf{q})(\sigma)}{(\theta\mathbf{p})(\sigma)}\right) \quad (28)$$

$$= \sum_{\sigma} \theta(\sigma, \cdot) \mathbf{p} f\left(\frac{\theta(\sigma, \cdot) \mathbf{q}}{\theta(\sigma, \cdot) \mathbf{p}}\right) \quad (29)$$

$$= \sum_{\sigma} \theta(\sigma, \cdot) \mathbf{p} f\left(\frac{1}{\theta(\sigma, \cdot) \mathbf{p}} \sum_{\sigma'} \theta(\sigma, \sigma') \mathbf{p}(\sigma') \frac{\mathbf{q}(\sigma')}{\mathbf{p}(\sigma')}\right) \quad (30)$$

$$\leq \sum_{\sigma} \theta(\sigma, \cdot) \mathbf{p} \frac{1}{\theta(\sigma, \cdot) \mathbf{p}} \sum_{\sigma'} \theta(\sigma, \sigma') \mathbf{p}(\sigma') f\left(\frac{\mathbf{q}(\sigma')}{\mathbf{p}(\sigma')}\right) \quad (31)$$

$$= \sum_{\sigma} \mathbf{p}(\sigma) f\left(\frac{\mathbf{q}(\sigma)}{\mathbf{p}(\sigma)}\right) = D_f(\mathbf{p}, \mathbf{q}) \quad (32)$$

The second equality holds since $(\theta\mathbf{p})(\sigma)$ is dot product of the σ^{th} row of θ and \mathbf{p} .

The third equality holds since $\sum_{\sigma'} \theta(\sigma, \sigma') \mathbf{p}(\sigma') \frac{\mathbf{q}(\sigma')}{\mathbf{p}(\sigma')} = \theta(\sigma, \cdot) \mathbf{q}$.

The fourth inequality follows from the convexity of $f(\cdot)$.

The last equality holds since $\sum_{\sigma} \theta(\sigma, \sigma') = 1$.

We now examine under what conditions the inequality in Equation 31 is strict. Note that for any strictly convex function g , if $\forall u, \lambda_u > 0$, $g(\sum_u \lambda_u x_u) = \sum_u \lambda_u g(x_u)$ if and only if there exists x such that $\forall u, x_u = x$. By this property, the inequality is strict if and only if there exists $\sigma, \sigma', \sigma''$ such that $\frac{\mathbf{q}(\sigma')}{\mathbf{p}(\sigma')} \neq \frac{\mathbf{q}(\sigma'')}{\mathbf{p}(\sigma'')}$ and $\theta(\sigma, \sigma')\mathbf{p}(\sigma') > 0$, $\theta(\sigma, \sigma'')\mathbf{p}(\sigma'') > 0$. \square

Lemma 8.8 (Prelec [12]). *Given agents play equilibrium \mathbf{s} , for every agent i ,*

$$\mathbb{E}(\text{score}_{Im}(r_i, r_j) | j, \mathbf{s}) \leq KL\text{-}IG_{\sigma_j, \sigma_i}^{\mathbf{s}}$$

where $\mathbb{E}(\text{score}_{Im}(r_i, r_j) | j, \mathbf{s})$ is the expected information score of agent i conditioning on her reference agent being agent j , and $KL\text{-}IG$ is the f -information gain when f -divergence is chosen as KL divergence.

The equality holds when everyone tells the truth.

Proof. In an equilibrium \mathbf{s} , based on the property of proper scoring rule, $\hat{q}_j(\hat{\sigma}_i) = \sum_{\hat{\sigma}_{-ij}} fr(\hat{\sigma}_i|\hat{\sigma}_{-ij})Pr(\hat{\sigma}_{-ij}|\sigma_j)$.

When the number of agents is infinite, the fraction of $\hat{\sigma}_i$ among $\hat{\sigma}_{-i}$ or $\hat{\sigma}_{-j}$ equals that among $\hat{\sigma}_{-ij}$, that is, $fr(\hat{\sigma}_i|\hat{\sigma}_{-i}) = fr(\hat{\sigma}_i|\hat{\sigma}_{-j}) = fr(\hat{\sigma}_i|\hat{\sigma}_{-ij})$.

Now we define a probability measure W such that $W(\hat{\sigma}_i, \hat{\sigma}_{-ij}, \sigma_j) = Pr(\hat{\sigma}_{-ij}, \sigma_j) * fr(\hat{\sigma}_i|\hat{\sigma}_{-ij})$. Note that

$$W(\hat{\sigma}_{-ij}, \sigma_j) = Pr(\hat{\sigma}_{-ij}, \sigma_j)$$

With this new definition, we can rewrite $\hat{q}_j(\hat{\sigma}_i)$ as $W(\hat{\sigma}_i|\sigma_j)$ since

$$\hat{q}_j(\hat{\sigma}_i) = \sum_{\hat{\sigma}_{-ij}} fr(\hat{\sigma}_i|\hat{\sigma}_{-ij})Pr(\hat{\sigma}_{-ij}|\sigma_j) \quad (33)$$

$$= \sum_{\hat{\sigma}_{-ij}} \frac{W(\hat{\sigma}_i, \hat{\sigma}_{-ij}, \sigma_j)}{Pr(\sigma_j)} \quad (34)$$

$$= W(\hat{\sigma}_i|\sigma_j) \quad (35)$$

Thus we have

$$\log \frac{fr(\hat{\sigma}_i|\hat{\sigma}_{-i})}{\hat{q}_j(\hat{\sigma}_i)} = \log \frac{fr(\hat{\sigma}_i|\hat{\sigma}_{-ij})}{W(\hat{\sigma}_i|\sigma_j)}$$

Moreover,

$$\log \frac{fr(\hat{\sigma}_i|\hat{\sigma}_{-ij})}{W(\hat{\sigma}_i|\sigma_j)} \quad (36)$$

$$= \log \frac{W(\hat{\sigma}_i, \hat{\sigma}_{-ij}, \sigma_j)}{W(\hat{\sigma}_i|\sigma_j)Pr(\hat{\sigma}_{-ij}, \sigma_j)} \quad (37)$$

$$= \log \frac{W(\hat{\sigma}_i, \hat{\sigma}_{-ij}, \sigma_j)}{W(\hat{\sigma}_i|\sigma_j)W(\hat{\sigma}_{-ij}, \sigma_j)} \quad (38)$$

$$= \log \frac{W(\hat{\sigma}_i, \hat{\sigma}_{-ij}, \sigma_j)W(\sigma_j)}{W(\hat{\sigma}_i, \sigma_j)W(\hat{\sigma}_{-ij}, \sigma_j)} \quad (39)$$

$$= \log \frac{W(\hat{\sigma}_{-ij}|\hat{\sigma}_i, \sigma_j)}{W(\hat{\sigma}_{-ij}|\sigma_j)} \quad (40)$$

$$= \log \frac{W(\hat{\sigma}_{-ij}|\hat{\sigma}_i, \sigma_j)}{Pr(\hat{\sigma}_{-ij}|\sigma_j)} \quad (41)$$

Therefore, we have

$$\mathbb{E}(score_{Im}(r_i, r_j)|j) = \sum_{\hat{\sigma}_{-ij}} Pr(\hat{\sigma}_{-ij}|\sigma_i, \sigma_j) \log \frac{W(\hat{\sigma}_{-ij}|\hat{\sigma}_i, \sigma_j)}{Pr(\hat{\sigma}_{-ij}|\sigma_j)} \quad (42)$$

$$\leq \sum_{\hat{\sigma}_{-ij}} Pr(\hat{\sigma}_{-ij}|\sigma_i, \sigma_j) \log \frac{Pr(\hat{\sigma}_{-ij}|\sigma_i, \sigma_j)}{Pr(\hat{\sigma}_{-ij}|\sigma_j)} = KL-IG_{\sigma_j, \sigma_i}^{\mathbf{s}} \quad (43)$$

The inequality follows from the property of proper scoring rule. Note that when everyone tells the truth,

$$W(\hat{\sigma}_{-ij}, \hat{\sigma}_i, \sigma_j) = W(\sigma_{-ij}, \sigma_i, \sigma_j) \quad (44)$$

$$= Pr(\sigma_{-ij}, \sigma_j) * fr(\sigma_i|\sigma_{-ij}) \quad (45)$$

According the conditional independence assumption (see Assumption 8.4), when everyone tells the truth and the number of agents is infinite, σ_{-i} gives the state of world ω and $\omega(\sigma_i)$ is the fraction of σ_i in σ_{-i} . Thus

$$Pr(\sigma_i|\sigma_{-ij}, \sigma_j) = fr(\sigma_i|\sigma_{-ij})$$

we have

$$W(\hat{\sigma}_{-ij}, \hat{\sigma}_i, \sigma_j) = W(\sigma_{-ij}, \sigma_i, \sigma_j) \tag{46}$$

$$= Pr(\sigma_{-ij}, \sigma_j) * Pr(\sigma_i|\sigma_{-ij}, \sigma_j) \tag{47}$$

$$= Pr(\sigma_{-ij}, \sigma_i, \sigma_j) \tag{48}$$

Therefore, $W(\sigma_{-ij}|\sigma_i, \sigma_j) = Pr(\sigma_{-ij}|\sigma_i, \sigma_j)$ which implies that $\mathbb{E}(score_{Im}(r_i, r_j)|j) = KL-IG_{\sigma_j, \sigma_i}^{\mathbf{T}}$ when everyone tells the truth.

□